LOCAL WELL-POSEDNESS AND A PRIORI BOUNDS FOR THE MODIFIED BENJAMIN-ONO EQUATION WITHOUT USING A GAUGE TRANSFORMATION

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Abstract. We prove that the complex-valued modified Benjamin-Ono (mBO) equation is locally wellposed if the initial data ϕ belongs to H^s for $s \geq 1/2$ with $\|\phi\|_{L^2}$ sufficiently small without performing a gauge transformation. Hence the real-valued mBO equation is globally wellposed for those initial datas, which is contained in the results of C. Kenig and H. Takaoka [25] where the smallness condition is not needed. We also prove that the real-valued H^∞ solutions to mBO equation satisfy a priori local in time H^s bounds in terms of the H^s size of the initial data for $s > 1/4$.

Contents

1. Introduction

In this paper, we study the initial value problem for the (defocusing) modified Benjamin-Ono equation of the form (also the equation with focusing nonlinearity of the form $-u^2u_x$ can be treated by our method)

$$\begin{aligned} &u_t + \mathcal{H}u_{xx} = u^2u_x,\ (x,t) \in \mathbb{R}^2, \\ &u(x,0) = \phi(x), \end{aligned} \tag{1.1}$$

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where $u:\mathbb{R}^2\to\mathbb{C}$ is a complex-valued function and $\mathcal{H}$ is the Hilbert transform which is defined as following

$$\mathcal{H}u(x)=\frac{1}{\pi}\text{p.v.}\int_{-\infty}^{+\infty}\frac{u(y)}{x-y}dy. \tag{1.2}$$

The equation with quadratic nonlinearity

$$u_t+\mathcal{H}u_{xx}=uu_x \tag{1.3}$$

was derived by Benjamin [2] and Ono [31] as a model for one-dimensional waves in deep water. On the other hand, the cubic nonlinearity is also of much interest for long wave models [1, 19].

The initial value problems for (1.1) and for the Benjamin-Ono equation (1.3) have been extensively studied [3, 7, 9, 10, 11, 13, 17, 18, 20, 21, 25, 27, 28, 29, 30, 32]. For instance, the energy method provides the wellposedness on the Sobolev space H^s for $s>3/2$ [17]. For the Benjamin-Ono equation (1.3), it has been know [18, 27] that this is locally wellposed for $s>9/8$ by the refinement of the energy method and dispersive estimates. Tao [34] obtained the global wellposedness in H^s for $s\geq 1$ by performing a gauge transformation as for the derivative Schrödinger equation and using the conservation law. This result was improved by Ionescu and Kenig [14] who obtained global wellposedness for $s\geq 0$, and also by Burq and Planchon [4] who obtained local wellposedness for $s>1/4$.

For the modified Benjamin-Ono equation (1.1), Molinet and Ribaud [29] obtained the local wellposedness in Sobolev space H^s for $s>1/2$. Their proof is based on Tao's gauge transformation [34]. The result for $s=1/2$ has been obtained by Kenig and Takaoka [25] by using frequency dyadically localized gauge transformation. Their result was sharp in the sense that the solution map is not uniformly continuous in H^s for $s<1/2$. With the Sobolev space H^s replaced by the Besov space $B^s_{2,1}$, the result has been obtained in [30] under a smallness condition on the data. To the author's knowledge, these results are all restricted to the real-valued mBO equation where the gauge is easy to handle. Our method in this paper can also deal with the complex-valued mBO equation.

The mBO equation (1.1) has several symmetries. The first one is the scaling invariance

$$u(x,t)\to u_\lambda=\frac{1}{\lambda^{1/2}}u(\frac{x}{\lambda},\frac{t}{\lambda^2}),\quad \phi_\lambda=\frac{1}{\lambda^{1/2}}\phi(\frac{x}{\lambda}), \tag{1.4}$$

which leads to the constraint $s\geq 0$ on the wellposedness for (1.1). We see that the equation (1.1) is L^2 critical, hence the L^2 norm of the initial data is not automatically small by the scaling, which is the main reason for us assuming the initial data has a small L^2 norm. There are at least the following three conservation laws preserved under the flow of the real-valued mBO equation (1.1)

$$\frac{d}{dt}\int_{\mathbb{R}}u(x,t)dx=0, \tag{1.5}$$

$$\frac{d}{dt}\int_{\mathbb{R}}u(x,t)^2dx=0, \tag{1.6}$$

$$\frac{d}{dt}\int_{\mathbb{R}}\frac{1}{2}u\mathcal{H}u_x-\frac{1}{12}|u(x,t)|^4dx=0. \tag{1.7}$$

These conservation laws provide a priori bounds on the solution. For instance, we get from (1.6) and (1.7) that the $H^{1/2}$ norm of the solution remains bounded for

finite time if the initial data belongs to $H^{1/2}$. Thus once one obtains a solution of existence interval with a length determined by the $H^{1/2}$ norm of the initial data, then the solution is automatically extended to a global one.

In the first part of this paper, we reprove the results of Kenig and Takaoka [25] without using a gauge transformation, but under an extra condition that the L^2 norm of the initial data is small. Since we don't perform a gauge transformation, our proof also works for the complex-valued Cauchy problem (1.1). Our method is to use the standard fixed-point argument in some Banach space. Bourgain's space $X^{s,b}$ defined as a closure of the following space

$$\{f \in \mathcal{S}(\mathbb{R}^2) : \|f\|_{X^{s,b}} = \|\langle\xi\rangle^s \langle\tau - \omega(\xi)\rangle^b \widehat{f}(\xi,\tau)\|_{L^2}\}$$

is very useful in the study of the low regularity theory of the nonlinear dispersive equations [5, 22, 16]. One might try a direct perturbative approach in $X^{s,b}$ space as Kenig, Ponce and Vega [22] did for the KdV and modified KdV equations. However, one will find that the key trilinear estimate

$$\|\partial_x(u^3)\|_{X^{s,b-1}} \lesssim \|u\|^3_{X^{s,b}}, \quad \text{for some } b \in [1/2, 1) \tag{1.8}$$

fails for any s due to logarithmic divergences involving the modulation variable (see Proposition 5.7, 5.8 below). The key observation in this paper is that these logarithmic divergences can be removed by us using Banach spaces which combine $X^{s,b}$ structure with smoothing effect structure. The spaces of these structures were first found and used by Ionescu and Kenig [14] to remove some logarithmic divergence.

Theorem 1.1. *Let $s \geq 1/2$. Assume $u_0 \in H^s$ with $\|u_0\|_{L^2} \ll 1$. Then*

(a) Existence. There exists $T = T(\|u_0\|_{H^{1/2}}) > 0$ and a solution u to the complex-valued mBO equation (1.1) *(or its focusing version) satisfying*

$$u \in F^s(T) \subset C([-T,T] : H^s), \tag{1.9}$$

where the function space $F^s(T)$ will be defined later (see section 2).

(b) Uniqueness. For the real-valued case, the solution mapping $u_0 \to u$ is the unique extension of the mapping $H^\infty \to C([-T,T] : H^\infty)$. For the complex-valued case, the solution u is unique in $B(u_0)$ which is defined in (6.13).

(c) Lipschitz continuity. For any $R > 0$, the mapping $u_0 \to u$ is Lipschitz continuous from $\{u_0 \in H^s : \|u_0\|_{H^s} < R, \|u_0\|_{L^2} \ll 1\}$ to $C([-T,T] : H^s)$.

For the real-valued mBO equation, from the conservation laws (1.6), (1.7), and iterating Theorem 1.1, we obtain the following corollary.

Corollary 1.2. *The Cauchy problem for the real-valued mBO equation* (1.1) *(or its focusing version) is globally wellposed if ϕ belongs to H^s for $s \geq 1/2$ with $\|\phi\|_{L^2}$ sufficiently small*

In the second part, we study the low regularity problem of the real-valued mBO equation (1.1). From the ill-posedness result in [25], we see that for $s < 1/2$ one can't use a direct contraction mapping method, but we expect some wellposedness results hold in the weak sense.

Conjecture. *The solution map $S_T^\infty : H^\infty \to C([-T,T] : H^\infty)$ of the real-valued modified Benjamin-Ono equation* (1.1) *can be uniquely extended to a continuous map from H^s to $C([-T,T] : H^s)$ for a small $T = T(\|\phi\|_{H^s}) > 0$ if $s > 1/4$.*

To prove this one would need to establish a priori H^s bounds for the H^∞ solutions and then prove continuous dependence on the initial data. We solve the easier half of this problem.

Theorem 1.3. *Let $s > 1/4$. For any $M > 0$ there exists $T > 0$ and $C > 0$ so that for any initial data $u_0 \in H^\infty$ satisfying*

$$\|u_0\|_{H^s} \leq M$$

then the solutions $u \in C([0,T] : H^\infty)$ to the real-valued mBO equation (1.1) satisfies

$$\|u\|_{H^s} \leq C\|u_0\|_{H^s}. \tag{1.10}$$

Remark 1.4. A very similar equation to (1.1) is the derivative nonlinear Schrödinger equation

$$u_t - iu_{xx} = |u|^2 u_x, \quad (x,t) \in \mathbb{R}^2, \tag{1.11}$$

and local wellposedness was known for the equation in H^s for $s \geq 1/2$ [33], where a fixed point argument is performed in an adapted Bourgain's $X^{s,b}$ space after a gauge transformation on the equation. Our methods also give the same results in Theorem 1.1 and 1.3 for (1.11).

We discuss now some of the ingredients in the proof of Theorem 1.3. We will follow the method of Ionescu, Kenig and Tataru [16] which approaches the problem in a less perturbative way. It can be viewed as a combination of the energy method and the perturbative method. More precisely, we will define $F^{l,s}(T), N^{l,s}(T)$ and energy space $E^{l,s}(T)$ and show that if u is a smooth solution of (1.1) on $\mathbb{R} \times [-T,T]$ with $\|u\|_{E^{l,s}(T)} \ll 1$, then

$$\begin{cases} \|u\|_{F^{l,s}(T)} \lesssim \|u\|_{E^{l,s}(T)} + \|\partial_x(u^3)\|_{N^{l,s}(T)}; \\ \|\partial_x(u^3)\|_{N^{l,s}(T)} \lesssim \|u\|^3_{F^{l,s}(T)}; \\ \|u\|^2_{E^{l,s}(T)} \lesssim \|\phi\|^2_{\dot{H}^l \cap \dot{H}^s} + \|u\|^3_{F^{l,s}(T)}. \end{cases} \tag{1.12}$$

The inequalities (1.12) and a simple continuity argument still suffice to control $\|u\|_{F^{l,s}(T)}$, provided that $\|\phi\|_{\dot{H}^l \cap \dot{H}^s} \ll 1$ (which can by arranged by rescaling if $l, s > 0$). The first inequality in (1.12) is the analogue of the linear estimate. The second inequality in (1.12) is the analogue of the trilinear estimate (1.8). The last inequality in (1.12) is an energy-type estimate.

We explain the strategies in [16] to define the main normed and semi-normed spaces. As was explained before, standard using of $X^{s,b}$ spaces for fixed-point argument will lead to a logarithmic divergence in the key trilinear estimate. But we use $X^{s,b}$-type structures only on small, frequency dependant time intervals. The high-low interaction can be controlled for short time. The second step is to define $\|u\|_{E^{l,s}(T)}$ sufficiently large to be able to still prove the linear estimate $\|u\|_{F^{l,s}(T)} \lesssim \|u\|_{E^{l,s}(T)} + \|\partial_x(u^3)\|_{N^{l,s}(T)}$. Finally, we use frequence-localized energy estimates and the symmetries of the equation (1.1) to prove the energy estimate $\|u\|^2_{E^{l,s}(T)} \lesssim \|\phi\|^2_{\dot{H}^l \cap \dot{H}^s} + \|u\|^3_{F^{l,s}(T)}$.

The rest of the paper is organized as follows: In section 2 we present some notations and Banach function spaces. We summarize some properties of the spaces in section 3. A symmetric estimate will be given in section 4 which is used in section 5 to show dyadic trilinear estimate. The proof of Theorem 1.1 is given in section 6. In section 7 we prove short time dyadic trilinear estimate and in section 8 we prove an energy estimate. Finally in section 9 we prove Theorem 1.3.

2. Notation and Definitions

For $x, y \in \mathbb{R}^+$, $x \lesssim y$ means that there exists $C > 0$ such that $x \leq Cy$. By $x \sim y$ we mean $x \lesssim y$ and $y \lesssim x$. For $f \in \mathcal{S}'$ we denote by $\widehat{f}$ or $\mathcal{F}(f)$ the Fourier transform of f for both spatial and time variables,

$$\widehat{f}(\xi, \tau) = \int_{\mathbb{R}^2} e^{-ix\xi} e^{-it\tau} f(x,t) dxdt.$$

Besides, we use $\mathcal{F}_x$ and $\mathcal{F}_t$ to denote the Fourier transform with respect to space and time variable respectively. Let $\mathbb{Z}_+ = \mathbb{Z} \cap [0, \infty)$. For $k \in \mathbb{Z}$ let

$$O_k = \{\xi : |\xi| \in [(3/4) \cdot 2^k, (3/2) \cdot 2^k)\ \} \text{ and } I_k = \{\xi : |\xi| \in [2^{k-1}, 2^{k+1}]\ \}.$$

For $k \in \mathbb{Z}_+$ let $\widetilde{I}_k = [-2, 2]$ if $k = 0$ and $\widetilde{I}_k = I_k$ if $k \geq 1$.

Let $\eta_0 : \mathbb{R} \to [0, 1]$ denote an even smooth function supported in $[-8/5, 8/5]$ and equal to 1 in $[-5/4, 5/4]$. For $k \in \mathbb{Z}$ let $\chi_k(\xi) = \eta_0(\xi/2^k) - \eta_0(\xi/2^{k-1})$, χ_k supported in $\{\xi : |\xi| \in [(5/8) \cdot 2^k, (8/5) \cdot 2^k]\}$, and

$$\chi_{[k_1,k_2]} = \sum_{k=k_1}^{k_2} \chi_k \text{ for any } k_1 \leq k_2 \in \mathbb{Z}.$$

For simplicity of notation, let $\eta_k = \chi_k$ if $k \geq 1$ and $\eta_k \equiv 0$ if $k \leq -1$. Also, for $k_1 \leq k_2 \in \mathbb{Z}$ let

$$\eta_{[k_1,k_2]} = \sum_{k=k_1}^{k_2} \eta_k \text{ and } \eta_{\leq k_2} = \sum_{k=-\infty}^{k_2} \eta_k.$$

Roughly speaking, $\{\chi_k\}_{k\in\mathbb{Z}}$ is the homogeneous decomposition function sequence and $\{\eta_k\}_{k\in\mathbb{Z}_+}$ is the non-homogeneous decomposition function sequence to the frequency space.

For $k \in \mathbb{Z}$ let $k_+ = \max(k, 0)$, and let P_k, R_k denote the operators on $L^2(\mathbb{R})$ defined by

$$\widehat{P_k u}(\xi) = \chi_k(\xi)\widehat{u}(\xi) \text{ and } \widehat{R_k u}(\xi) = 1_{O_k}(\xi)\widehat{u}(\xi).$$

By a slight abuse of notation we also define the operators P_k, R_k on $L^2(\mathbb{R} \times \mathbb{R})$ by formulas $\mathcal{F}(P_k u)(\xi, \tau) = \chi_k(\xi)\mathcal{F}(u)(\xi, \tau)$ and $\mathcal{F}(R_k u)(\xi, \tau) = 1_{O_k}(\xi)\mathcal{F}(u)(\xi, \tau)$. For $l \in \mathbb{Z}$ let

$$P_{\leq l} = \sum_{k \leq l} P_k, \quad P_{\geq l} = \sum_{k \geq l} P_k.$$

Similarly we also define the operators $R_{\leq l}$ and $R_{\geq l}$.

Let $a_1, a_2, a_3, a_4 \in \mathbb{R}$. It will be convenient to define the quantities $a_{max} \geq a_{sub} \geq a_{thd} \geq a_{min}$ to be the maximum, sub-maximum, third-maximum, and minimum of a_1, a_2, a_3, a_4 respectively. We also denote $\mathrm{sub}(a_1, a_2, a_3, a_4) = a_{sub}$ and $\mathrm{thd}(a_1, a_2, a_3, a_4) = a_{thd}$. Usually we use k_1, k_2, k_3, k_4 and j_1, j_2, j_3, j_4 to denote integers, $N_i = 2^{k_i}$ and $L_i = 2^{j_i}$ for $i = 1, 2, 3, 4$ to denote dyadic numbers. For $a \in \mathbb{R}$ we define $a-$ to be the real number $a - \epsilon$ for some $0 < \epsilon \ll 1$. Similar we also define $a+$.

For $\xi \in \mathbb{R}$ let

$$\omega(\xi) = -\xi|\xi|, \tag{2.1}$$

which is the dispersion relation associated to the linear Benjamin-Ono equation. For $\phi \in L^2(\mathbb{R})$ let $W(t)\phi \in C(\mathbb{R} : L^2)$ denote the solution of the free Benjamin-Ono

evolution given by

$$\mathcal{F}_x[W(t)\phi](\xi,t)=e^{it\omega(\xi)}\widehat{\phi}(\xi), \tag{2.2}$$

where $\omega(\xi)$ is defined in (2.1). For $k\in\mathbb{Z}_+$ and $j\geq 0$ let $D_{k,j}=\{(\xi,\tau)\in\mathbb{R}\times\mathbb{R}: \xi\in\widetilde{I}_k,\tau-\omega(\xi)\in\widetilde{I}_j\}$. For $k\in\mathbb{Z}$ and $j\geq 0$ let $\dot{D}_{k,j}=\{(\xi,\tau)\in\mathbb{R}\times\mathbb{R}:\xi\in I_k,\tau-\omega(\xi)\in\widetilde{I}_j\}$. For $k\in\mathbb{Z}_+$ we define now the Banach spaces $X_k(\mathbb{R}\times\mathbb{R})$:

$$\begin{aligned} X_k &= \{f\in L^2(\mathbb{R}^2): f \text{ is supported in } \widetilde{I}_k\times\mathbb{R} \text{ and} \\ &\quad \|f\|_{X_k}:=\sum_{j=0}^{\infty}2^{j/2}\beta_{k,j}\|\eta_j(\tau-w(\xi))\cdot f(\xi,\tau)\|_{L^2_{\xi,\tau}}<\infty\}, \end{aligned} \tag{2.3}$$

where

$$\beta_{k,j}=1+2^{(j-2k)/2}. \tag{2.4}$$

Here the spaces X_k is the same as those used by Ionescu and Kenig [14] for $k>0$. X_0 is different, since we don't have the special structures in the low frequency. The precise choice of the coefficients $\beta_{k,j}$ is important in order for all the trilinear estimates in Section 5 to hold. We see that if k is small then $\beta_{k,j}\approx 2^{j/2}$. This factor is particularly important in controlling the high-low interaction. From the technical level, we know from the K-Z method of Tao [35] that the worst interaction is that the low frequency component has a largest modulation. But the factor $\beta_{k,j}$ will compensate for that. The logarithmic divergence caused by the other interaction, namely high frequency component with largest modulation, can be removed by using a smoothing effect structure.

As in [14], the spaces X_k are not sufficient for our purpose, due to various logarithmic divergences involving the modulation variable (See Proposition 5.7 below). For $k\geq 100$ we also define the Banach spaces $Y_k=Y_k(\mathbb{R}^2)$. For $k\geq 100$ we define

$$\begin{aligned} Y_k &= \{f\in L^2(\mathbb{R}^2): f \text{ is supported in } \bigcup_{j=0}^{k-1}D_{k,j} \text{ and} \\ &\quad \|f\|_{Y_k}:=2^{-k/2}\|\mathcal{F}^{-1}[(\tau-\omega(\xi)+i)f(\xi,\tau)]\|_{L^1_xL^2_t}<\infty\}. \end{aligned} \tag{2.5}$$

Then for $k\in\mathbb{Z}_+$ we define

$$Z_k:=X_k \text{ if } k\leq 99 \text{ and } Z_k:=X_k+Y_k \text{ if } k\geq 100. \tag{2.6}$$

The spaces Z_k are our basic Banach spaces. For $s\geq 0$ we define the Banach spaces $F^s=F^s(\mathbb{R}\times\mathbb{R})$:

$$F^s=\{u\in\mathcal{S}'(\mathbb{R}\times\mathbb{R}):\|u\|_{F^s}^2=\sum_{k=0}^{\infty}2^{2sk}\|\eta_k(\xi)\mathcal{F}(u)\|_{Z_k}^2<\infty\}, \tag{2.7}$$

and $N^s=N^s(\mathbb{R}\times\mathbb{R})$ which is used to measure the nonlinear term and can be viewed as an analogue of $X^{s,b-1}$

$$\begin{aligned} N^s &= \{u\in\mathcal{S}'(\mathbb{R}\times\mathbb{R}): \\ &\quad \|u\|_{N^s}^2=\sum_{k=0}^{\infty}2^{2sk}\|\eta_k(\xi)(\tau-\omega(\xi)+i)^{-1}\mathcal{F}(u)\|_{Z_k}^2<\infty\}. \end{aligned} \tag{2.8}$$

The spaces F^s and N^s have the same structures in high frequency as those in [14], but with different structures in low frequency.

In order to prove a priori bounds, we will need another set of norms and semi-norms which were first used by Ionescu, Kenig and Tataru in [16] for the KP-I equation. Similar idea can be found in [26]. For $k \in \mathbb{Z}$ we define

$$\begin{aligned} B_k &= \{f \in L^2(\mathbb{R}^2) : f \text{ is supported in } I_k \times \mathbb{R} \text{ and} \\ &\qquad \|f\|_{B_k} := \sum_{j=0}^{\infty} 2^{j/2} \|\eta_j(\tau - w(\xi)) \cdot f(\xi,\tau)\|_{L^2_{\xi,\tau}} < \infty\}. \end{aligned} \tag{2.9}$$

These l^1-type $X^{s,b}$ structures were first introduced and used in [37, 14, 16]. It is also useful in the study of uniform global wellposedness and inviscid limit for the KdV-Burgers equation [12].

At frequency 2^k we will use the $X^{s,b}$ structure given by the B_k norm, uniformly on the 2^{-k_+} time scale. For $k \in \mathbb{Z}$ we define the normed spaces

$$F_k = \left\{ \begin{array}{l} f \in L^2(\mathbb{R}^2) : \widehat{f} \text{ is supported in } I_k \times \mathbb{R} \text{ and} \\ \|f\|_{F_k} = \sup_{t_k \in \mathbb{R}} \|\mathcal{F}[f \cdot \eta_0(2^{k_+}(t - t_k))]\|_{B_k} < \infty \end{array} \right\},$$

$$N_k = \left\{ \begin{array}{l} f \in L^2(\mathbb{R}^2) : \widehat{f} \text{ is supported in } I_k \times \mathbb{R} \text{ and } \|f\|_{N_k} = \\ \sup_{t_k \in \mathbb{R}} \|(\tau - \omega(\xi) + i2^{k_+})^{-1} \mathcal{F}[f \cdot \eta_0(2^{k_+}(t - t_k))]\|_{B_k} < \infty \end{array} \right\}.$$

The bounds we obtain for smooth solutions of the equation (1.1) are on a fixed time interval, while the above function spaces are not. Thus we define a local version of the spaces. For $T \in (0, 1]$ we define the normed spaces

$$\begin{aligned} F_k(T) &= \{f \in C([-T,T] : E_k) : \|f\|_{F_k(T)} = \inf_{\widetilde{f}=f \text{ in } \mathbb{R}\times[-T,T]} \|\widetilde{f}\|_{F_k}\}; \\ N_k(T) &= \{f \in C([-T,T] : E_k) : \|f\|_{N_k(T)} = \inf_{\widetilde{f}=f \text{ in } \mathbb{R}\times[-T,T]} \|\widetilde{f}\|_{N_k}\}. \end{aligned}$$

For $l, s \geq 0$ and $T \in (0, 1]$, we define the normed spaces

$$F^{l,s}(T) = \left\{ \begin{array}{l} u \in C([-T,T] : H^\infty) : \quad \|u\|^2_{F^{l,s}} = \\ \sum_{k=-\infty}^{0} 2^{2lk} \|R_k(u)\|^2_{F_k(T)} + \sum_{k=1}^{\infty} 2^{2sk} \|R_k(u)\|^2_{F_k(T)} < \infty \end{array} \right\},$$

$$N^{l,s}(T) = \left\{ \begin{array}{l} u \in C([-T,T] : H^\infty) : \quad \|u\|^2_{N^{l,s}} = \\ \sum_{k=-\infty}^{0} 2^{2lk} \|R_k(u)\|^2_{N_k(T)} + \sum_{k=1}^{\infty} 2^{2sk} \|R_k(u)\|^2_{N_k(T)} < \infty \end{array} \right\}.$$

We still need an energy space. For $l, s \geq 0$ and $u \in C([-T,T] : H^\infty)$ we define

$$\|u\|^2_{E^{l,s}(T)} = \|R_{\leq 0}(u(0))\|^2_{\dot{H}^l} + \sum_{k \geq 1} \sup_{t_k \in [-T,T]} 2^{2sk} \|R_k(u(t_k))\|^2_{L^2}.$$

3. Properties of the Spaces Z_k

In this section we devote to study the properties of the spaces Z_k. Using the definitions, if $k \in \mathbb{Z}_+$ and $f_k \in Z_k$ then f_k can be written in the form

$$\left\{ \begin{array}{l} f_k = \sum_{j=0}^{\infty} f_{k,j} + g_k; \\ \sum_{j=0}^{\infty} 2^{j/2} \beta_{k,j} \|f_{k,j}\|_{L^2} + \|g_k\|_{Y_k} \leq 2\|f_k\|_{Z_k}, \end{array} \right. \tag{3.1}$$

such that $f_{k,j}$ is supported in $D_{k,j}$ and g_k is supported in $\cup_{j=0}^{k-1} D_{k,j}$ (if $k \leq 99$ then $g_k \equiv 0$). We start with the elementary properties.

Lemma 3.1 (Lemma 4.1, [14])**.** *(a) If $m, m' : \mathbb{R} \to \mathbb{C}$, $k \in \mathbb{Z}_+$, and $f_k \in Z_k$ then*

$$\left\{ \begin{array}{l} \|m(\xi) f_k(\xi,\tau)\|_{Z_k} \leq C \|\mathcal{F}^{-1}(m)\|_{L^1(\mathbb{R})} \|f_k\|_{Z_k}; \\ \|m'(\tau) f_k(\xi,\tau)\|_{Z_k} \leq C \|m\|_{L^\infty(\mathbb{R})} \|f_k\|_{Z_k}. \end{array} \right. \tag{3.2}$$

(b) If $k \in \mathbb{Z}_+$, $j \geq 0$, and $f_k \in Z_k$ then

$$\|\eta_j(\tau - \omega(\xi)) f_k(\xi, \tau)\|_{X_k} \leq C\|f_k\|_{Z_k}. \tag{3.3}$$

(c) If $k \geq 1$, $j \in [0, k]$, and f_k is supported in $I_k \times \mathbb{R}$ then

$$\|\mathcal{F}^{-1}[\eta_{\leq j}(\tau - \omega(\xi)) f_k(\xi, \tau)]\|_{L^1_x L^2_t} \leq C\|\mathcal{F}^{-1}(f_k)\|_{L^1_x L^2_t}. \tag{3.4}$$

We study now the embedding properties of the spaces Z_k. We will see that the spaces X_k and Y_k are very close.

Lemma 3.2. *Let $k \in \mathbb{Z}_+$, $s \in \mathbb{R}$ and $I \subset \mathbb{R}$ be an interval. Let Y be $L^p_x L^q_{t\in I}$ or $L^q_{t\in I} L^p_x$ for some $1 \leq p, q \leq \infty$ with the property that*

$$\|e^{-t\mathcal{H}\partial_x^2} f\|_Y \lesssim 2^{ks}\|f\|_{L^2(\mathbb{R})}$$

for all $f \in L^2(\mathbb{R})$ with $\widehat{f}$ supported in $\widetilde{I}_k$ and $\tau_0 \in \mathbb{R}$. Then we have that if $f_k \in Z_k$

$$\|\mathcal{F}^{-1}(f_k)\|_Y \lesssim 2^{ks}\|f_k\|_{Z_k}. \tag{3.5}$$

Proof. We assume first that $f_k = f_{k,j}$ with $\|f_k\|_{X_k} = 2^{j/2}\beta_{k,j}\|f_{k,j}\|_{L^2}$ and $f_{k,j}$ is supported in $D_{k,j}$ for some $j \geq 0$. Then we have

$$\begin{aligned}
\mathcal{F}^{-1}(f_k)(x,t) &= \int f_{k,j}(\xi,\tau) e^{ix\xi} e^{it\tau} d\xi d\tau \\
&= \int_{\widetilde{I}_j} e^{it\tau} \int f_{k,j}(\xi, \tau + \omega(\xi)) e^{ix\xi} e^{it\omega(\xi)} d\xi d\tau.
\end{aligned}$$

From the hypothesis on Y, we obtain

$$\begin{aligned}
\|\mathcal{F}^{-1}(f_k)(x,t)\|_Y &\lesssim \int \eta_j(\tau) \left\| e^{it\tau} \int f_{k,j}(\xi, \tau + \omega(\xi)) e^{ix\xi} e^{it\omega(\xi)} d\xi \right\|_Y d\tau \\
&\lesssim 2^{ks} 2^{j/2} \|f_{k,j}\|_{L^2},
\end{aligned}$$

which completes the proof in this case.

We assume now that $k \geq 100$ and $f_k = g_k \in Y_k$. From definition g_k can be written in the form

$$\begin{cases} g_k(\xi,\tau) = 2^{k/2}\chi_{[k-1,k+1]}(\xi)(\tau - \omega(\xi) + i)^{-1}\eta_{\leq k}(\tau - \omega(\xi))\mathcal{F}_x h(\xi,\tau); \\ \|g_k\|_{Y_k} = C\|h\|_{L^1_x L^2_\tau}. \end{cases} \tag{3.6}$$

It suffices to prove that if

$$f(\xi,\tau) = 2^{k/2}\chi_{[k-1,k+1]}(\xi)(\tau - \omega(\xi) + i)^{-1}\eta_{\leq k}(\tau - \omega(\xi)) \cdot h(\tau)$$

then

$$\left\| \int_{\mathbb{R}^2} f(\xi,\tau) e^{ix\xi} e^{it\tau} d\xi d\tau \right\|_Y \lesssim 2^{ks}\|h\|_{L^2}, \tag{3.7}$$

which follows from the proof of Lemma 4.2 (b) in [14]. ■

In order to obtain the more specific embedding properties of the spaces Z_k, we need the estimate for the free Benjamin-Ono equation. We recall the Strichartz estimates, smoothing effects, and maximal function estimates (for the proof, see, e.g. [24, 23] and the reference therein)

Lemma 3.3. *Let $k \in \mathbb{Z}_+$ and $I \subset \mathbb{R}$ be an interval with $|I|\lesssim 1$. Then for all $\phi \in L^2(\mathbb{R})$ with $\widehat{\phi}$ supported in $\widetilde{I}_k$,*

(a) Strichartz estimates

$$\|e^{-t\mathcal{H}\partial_x^2}\phi\|_{L_t^q L_x^r} \le C\|\phi\|_{L^2(\mathbb{R})},$$

where (q,r) is admissible, namely $2 \le q, r \le \infty$ and $2/q = 1/2 - 1/r$.

(b) Smoothing effect

$$\|e^{-t\mathcal{H}\partial_x^2}\phi\|_{L_x^\infty L_t^2} \le C2^{-k/2}\|\phi\|_{L^2(\mathbb{R})}.$$

(c) Maximal function estimate

$$\begin{aligned}\|e^{-t\mathcal{H}\partial_x^2}\phi\|_{L_x^2 L_{t\in I}^\infty} &\le C2^{k/2}\|\phi\|_{L^2(\mathbb{R})},\\ \|e^{-t\mathcal{H}\partial_x^2}\phi\|_{L_x^4 L_t^\infty} &\le C2^{k/4}\|\phi\|_{L^2(\mathbb{R})}.\end{aligned}$$

In particular we note the case $(6,6)$ is admissible which we will use in the sequel. From Lemma 3.2, 3.3, we immediately get the following

Lemma 3.4. *Let $k \in \mathbb{Z}_+$ and $I \subset \mathbb{R}$ be an interval with $|I|\lesssim 1$. Assume (q,r) is admissible and $f_k \in Z_k$. Then*

$$\begin{aligned}&\|\mathcal{F}^{-1}(f_k)\|_{L_x^2 L_{t\in I}^\infty} \le C2^{k/2}\|f_k\|_{Z_k}.\\ &\|\mathcal{F}^{-1}(f_k)\|_{L_x^\infty L_t^2} \le C2^{-k/2}\|f_k\|_{Z_k},\\ &\|\mathcal{F}^{-1}(f_k)\|_{L_x^4 L_t^\infty} \le C2^{k/4}\|f_k\|_{Z_k},\\ &\|\mathcal{F}^{-1}(f_k)\|_{L_t^q L_x^r} \le C\|f_k\|_{Z_k}.\end{aligned}$$

As a consequence,

$$F^s \subseteq C(\mathbb{R}; H^s) \text{ for any } s \ge 0. \tag{3.8}$$

Now we turn to study the properties of the space $F^{l,s}$. The definition shows easily that if $k \in \mathbb{Z}$ and $f_k \in B_k$ then

$$\left\|\int_{\mathbb{R}} |f_k(\xi,\tau')|d\tau'\right\|_{L_\xi^2} \lesssim \|f_k\|_{B_k}. \tag{3.9}$$

Moreover, if $k \in \mathbb{Z}$, $l \in \mathbb{Z}_+$, and $f_k \in B_k$ then

$$\begin{aligned}&\sum_{j=l+1}^{\infty} 2^{j/2}\left\|\eta_j(\tau-\omega(\xi))\cdot\int_{\mathbb{R}}|f_k(\xi,\tau')|\cdot 2^{-l}(1+2^{-l}|\tau-\tau'|)^{-4}d\tau'\right\|_{L^2}\\ &+2^{l/2}\left\|\eta_{\le l}(\tau-\omega(\xi))\cdot\int_{\mathbb{R}}|f_k(\xi,\tau')|\cdot 2^{-l}(1+2^{-l}|\tau-\tau'|)^{-4}d\tau'\right\|_{L^2}\\ &\lesssim \|f_k\|_{B_k}.\end{aligned} \tag{3.10}$$

In particular, if $k \in \mathbb{Z}$, $l \in \mathbb{Z}_+$, $t_0 \in \mathbb{R}$, $f_k \in B_k$, and $\gamma \in \mathcal{S}(\mathbb{R})$, then

$$\|\mathcal{F}[\gamma(2^l(t-t_0))\cdot\mathcal{F}^{-1}(f_k)]\|_{B_k}\lesssim\|f_k\|_{B_k}. \tag{3.11}$$

Indeed, to prove (3.10), first for the second term on the left-hand side of (3.10), we immediately get from Cauchy-Schwarz inequality and (3.9) that

$$\begin{aligned}&2^{l/2}\left\|\eta_{\le l}(\tau-\omega(\xi))\cdot\int_{\mathbb{R}}|f_k(\xi,\tau')|\cdot 2^{-l}(1+2^{-l}|\tau-\tau'|)^{-4}d\tau'\right\|_{L^2}\\ &\lesssim\left\|\int_{\mathbb{R}}|f_k(\xi,\tau')|d\tau'\right\|_{L_\xi^2}\lesssim\|f_k\|_{B_k}.\end{aligned}$$

For the first term on the left-hand side of (3.10), we decompose $f_k(\xi,\tau')=\sum_{j_1\geq 0} f_{k,j_1}$ where $f_{k,j_1}=f_k(\xi,\tau')\eta_{j_1}(\tau'-\omega(\xi))$, and then we get

$$\begin{aligned}
&\sum_{j=l+1}^{\infty} 2^{j/2}\left\|\eta_j(\tau-\omega(\xi))\cdot\int_{\mathbb{R}}|f_k(\xi,\tau')|\cdot 2^{-l}(1+2^{-l}|\tau-\tau'|)^{-4}d\tau'\right\|_{L^2}\\
\lesssim&\sum_{j=l+1}^{\infty}\sum_{j_1=0}^{\infty} 2^{j/2}\left\|\eta_j(\tau-\omega(\xi))\int_{\mathbb{R}}|f_{k,j_1}(\xi,\tau')|\cdot 2^{-l}(1+2^{-l}|\tau-\tau'|)^{-4}d\tau'\right\|_{L^2}\\
=&\sum_{j=l+1}^{\infty}\Big(\sum_{j_1>j+5}+\sum_{j_1<j-5}+\sum_{j-5\leq j_1\leq j+5}\Big)\\
&2^{j/2}\left\|\eta_j(\tau-\omega(\xi))\int_{\mathbb{R}}|f_{k,j_1}(\xi,\tau')|\cdot 2^{-l}(1+2^{-l}|\tau-\tau'|)^{-4}d\tau'\right\|_{L^2}\\
=:&\, I+II+III.
\end{aligned}$$

For the contribution of I, we first observe that $|\tau-\tau'|\sim 2^{j_1}$ in this case. Then we get that

$$I\lesssim\sum_{j_1\geq l}\sum_{j\leq j_1} 2^j 2^{3l}2^{-4j_1}\left\|\int_{\mathbb{R}}|f_{k,j_1}(\xi,\tau')|d\tau'\right\|_{L^2_\xi}\leq \|f_k\|_{B_k}.$$

Similarly we can estimate the contribution of II. For the third term III, using Young's inequality, then we get

$$III\lesssim\sum_{j=l+1}^{\infty}\sum_{|j-j_1|\leq 5} 2^{j/2}\|f_{k,j_1}\|_{L^2}\lesssim\|f_k\|_{B_k}.$$

As in [16], for any $k\in\mathbb{Z}$ we define the set S_k of $k-$*acceptable* time multiplication factors

$$S_k=\{m_k:\mathbb{R}\to\mathbb{R}:\|m_k\|_{S_k}=\sum_{j=0}^{10}2^{-jk_+}\|\partial^j m_k\|_{L^\infty}<\infty\}. \tag{3.12}$$

For instance, $\eta(2^{k_+}t)\in S_k$ for any η satisfies $\|\partial_x^k\eta\|_{L^\infty}\leq C$ for $j=1,2,\ldots,10$. Direct estimates using the definitions and (3.10) show that for any $s\geq 0$ and $T\in(0,1]$

$$\left\{\begin{array}{l}\|\sum_{k\in\mathbb{Z}}m_k(t)\cdot R_k(u)\|_{F^{l,s}(T)}\lesssim(\sup_{k\in\mathbb{Z}}\|m_k\|_{S_k})\cdot\|u\|_{F^{l,s}(T)};\\ \|\sum_{k\in\mathbb{Z}}m_k(t)\cdot R_k(u)\|_{N^{l,s}(T)}\lesssim(\sup_{k\in\mathbb{Z}}\|m_k\|_{S_k})\cdot\|u\|_{N^{l,s}(T)};\\ \|\sum_{k\in\mathbb{Z}}m_k(t)\cdot R_k(u)\|_{E^{l,s}(T)}\lesssim(\sup_{k\in\mathbb{Z}}\|m_k\|_{S_k})\cdot\|u\|_{E^{l,s}(T)}.\end{array}\right. \tag{3.13}$$

Actually, for instance we show the first inequality in (3.13). In view of definition, it suffices to prove that if $u_k\in F_k$, then

$$\|m_k(t)u_k\|_{F_k}\lesssim\|u_k\|_{F_k}\|m_k\|_{S_k},\quad \forall\ k\in\mathbb{Z}.$$

From (3.11) we see that we only need to prove that

$$\big|\mathcal{F}\big[m_k(\cdot)\eta_0(2^{k_+}(\cdot-t_k))\big]\big|\lesssim 2^{-k_+}(1+2^{-k_+}|\tau|)^{-4}\|m_k\|_{S_k},$$

which follows from partial integration and the definition of S_k.

4. A Symmetric Estimate

In this section we prove a symmetric estimate which will be used to prove a trilinear estimate. For $\xi_1, \xi_2, \xi_3 \in \mathbb{R}$ and $\omega : \mathbb{R} \to \mathbb{R}$ as in (2.1) let

$$\Omega(\xi_1, \xi_2, \xi_3) = \omega(\xi_1) + \omega(\xi_2) + \omega(\xi_3) - \omega(\xi_1 + \xi_2 + \xi_3). \tag{4.1}$$

This is the resonance function that plays a crucial role in the trilinear estimate of the $X^{s,b}$-type space. See [35] for a perspective discussion. For compactly supported functions $f, g, h, u \in L^2(\mathbb{R} \times \mathbb{R})$ let

$$\begin{aligned} J(f,g,h,u) = \int_{\mathbb{R}^6} & f(\xi_1,\mu_1)g(\xi_2,\mu_2)h(\xi_3,\mu_3) \\ & u(\xi_1+\xi_2+\xi_3, \mu_1+\mu_2+\mu_3+\Omega(\xi_1,\xi_2,\xi_3))d\xi_1 d\xi_2 d\xi_3 d\mu_1 d\mu_2 d\mu_3. \end{aligned}$$

Lemma 4.1. *Assume $k_1, k_2, k_3, k_4 \in \mathbb{Z}$ and $k_1 \le k_2 \le k_3 \le k_4$, $j_1, j_2, j_3, j_4 \in \mathbb{Z}_+$, and $f_{k_i,j_i} \in L^2(\mathbb{R} \times \mathbb{R})$ are nonnegative functions supported in $I_{k_i} \times \cup_{l=0}^{j_i} \widetilde{I}_l$, $i = 1, 2, 3, 4$. For simplicity we write $J = |J(f_{k_1,j_1}, f_{k_2,j_2}, f_{k_3,j_3}, f_{k_4,j_4})|$. Then*

(a) For any $k_1 \le k_2 \le k_3 \le k_4$ and $j_1, j_2, j_3, j_4 \in \mathbb{Z}_+$,

$$J \le C2^{(j_{min}+j_{thd})/2} 2^{(k_{min}+k_{thd})/2} \prod_{i=1}^{4} \|f_{k_i,j_i}\|_{L^2}. \tag{4.2}$$

(b) If $k_2 \le k_3 - 5$ and $j_2 \ne j_{max}$,

$$J \le C2^{(j_1+j_2+j_3+j_4)/2} 2^{-j_{max}/2} 2^{-k_{max}/2} 2^{k_{min}/2} \prod_{i=1}^{4} \|f_{k_i,j_i}\|_{L^2}; \tag{4.3}$$

if $k_2 \le k_3 - 5$ and $j_2 = j_{max}$,

$$J \le C2^{(j_1+j_2+j_3+j_4)/2} 2^{-j_{max}/2} 2^{-k_{max}/2} 2^{k_{thd}/2} \prod_{i=1}^{4} \|f_{k_i,j_i}\|_{L^2}. \tag{4.4}$$

(c) For any $k_1, k_2, k_3, k_4 \in \mathbb{Z}$ and $j_1, j_2, j_3, j_4 \in \mathbb{Z}_+$,

$$J \le C2^{(j_1+j_2+j_3+j_4)/2} 2^{-j_{max}/2} \prod_{i=1}^{4} \|f_{k_i,j_i}\|_{L^2}. \tag{4.5}$$

(d) If $k_{min} \le k_{max} - 10$, then

$$J \le C2^{(j_1+j_2+j_3+j_4)/2} 2^{-k_{max}} \prod_{i=1}^{4} \|f_{k_i,j_i}\|_{L^2}. \tag{4.6}$$

Proof. Let $A_{k_i}(\xi) = [\int_{\mathbb{R}} |f_{k_i,j_i}(\xi,\mu)|^2 d\mu]^{1/2}$, $i = 1, 2, 3, 4$. Using the Cauchy-Schwarz inequality and the support properties of the functions f_{k_i,j_i},

$$\begin{aligned} & |J(f_{k_1,j_1}, f_{k_2,j_2}, f_{k_3,j_3}, f_{k_4,j_4})| \\ \le\ & C2^{(j_{min}+j_{thd})/2} \int_{\mathbb{R}^3} A_{k_1}(\xi_1) A_{k_2}(\xi_1) A_{k_3}(\xi_1) A_{k_4}(\xi_1+\xi_2+\xi_3) d\xi_1 d\xi_2 d\xi_3 \\ \le\ & C2^{(k_{min}+k_{thd})/2} 2^{(j_{min}+j_{thd})/2} \prod_{i=1}^{4} \|f_{k_i,j_i}\|_{L^2}, \end{aligned}$$

which is part (a), as desired.

For part (b), in view of the support properties of the functions, it is easy to see that $J(f_{k_1,j_1}, f_{k_2,j_2}, f_{k_3,j_3}, f_{k_4,j_4}) \equiv 0$ unless

$$k_4 \le k_3 + 5. \tag{4.7}$$

Simple changes of variables in the integration and the observation that the function ω is odd show that

$$|J(f,g,h,u)| = |J(g,f,h,u)| = |J(f,h,g,u)| = |J(\widetilde{f},g,u,h)|,$$

where $\widetilde{f}(\xi,\mu) = f(-\xi,-\mu)$. We assume first that $j_2 \ne j_{max}$. Then we have several cases: if $j_4 = j_{max}$, then we will prove that if $g_i : \mathbb{R} \to \mathbb{R}_+$ are L^2 functions supported in I_{k_i}, $i = 1,2,3$, and $g : \mathbb{R}^2 \to \mathbb{R}_+$ is an L^2 function supported in $I_{k_4} \times \widetilde{I}_{j_4}$, then

$$\begin{aligned}&\int_{\mathbb{R}^3} g_1(\xi_1)g_2(\xi_2)g_3(\xi_3)g(\xi_1+\xi_2+\xi_3, \Omega(\xi_1,\xi_2,\xi_3))d\xi_1 d\xi_2 d\xi_3\\ &\lesssim 2^{-k_{max}/2}2^{k_{min}/2}\|g_1\|_{L^2}\|g_2\|_{L^2}\|g_3\|_{L^2}\|g\|_{L^2}.\end{aligned} \tag{4.8}$$

This suffices for (4.3).

To prove (4.8), we first observe that since $k_2 \le k_3 - 5$ then $|\xi_3+\xi_2| \sim |\xi_3|$. By change of variable $\xi_1' = \xi_1$, $\xi_2' = \xi_2$, $\xi_3' = \xi_3 + \xi_2$, we get that the left side of (4.8) is bounded by

$$\begin{aligned}&\int_{|\xi_1|\sim 2^{k_1}, |\xi_2|\sim 2^{k_2}, |\xi_3|\sim 2^{k_3}} g_1(\xi_1)g_2(\xi_2)\\ &g_3(\xi_3-\xi_2)g(\xi_1+\xi_3, \Omega(\xi_1,\xi_2,\xi_3-\xi_2))d\xi_1 d\xi_2 d\xi_3.\end{aligned} \tag{4.9}$$

Note that in the integration area we have

$$\Big|\frac{\partial}{\partial_{\xi_2}}\left[\Omega(\xi_1,\xi_2,\xi_3-\xi_2)\right]\Big| = |\omega'(\xi_2) - \omega'(\xi_3-\xi_2)| \sim 2^{k_3},$$

where we use the fact $\omega'(\xi) = |\xi|$ and $k_2 \le k_3 - 5$. By change of variable $\mu_2 = \Omega(\xi_1,\xi_2,\xi_3-\xi_2)$, we get that (4.9) is bounded by

$$\begin{aligned}&2^{-k_3/2}\int_{|\xi_1|\sim 2^{k_1}} g_1(\xi_1)\|g_2\|_{L^2}\|g_3\|_{L^2}\|g\|_{L^2}d\xi_1\\ \lesssim\ &2^{-k_{max}/2}2^{k_{min}/2}\|g_1\|_{L^2}\|g_2\|_{L^2}\|g_3\|_{L^2}\|g_1\|_{L^2}\|g\|_{L^2}.\end{aligned} \tag{4.10}$$

If $j_3 = j_{max}$, this case is identical to the case $j_4 = j_{max}$ in view of (4.7). If $j_1 = j_{max}$ it suffices to prove that if $g_i : \mathbb{R} \to \mathbb{R}_+$ are L^2 functions supported in I_{k_i}, $i = 2,3,4$, and $g : \mathbb{R}^2 \to \mathbb{R}_+$ is an L^2 function supported in $I_{k_1} \times \widetilde{I}_{j_1}$, then

$$\begin{aligned}&\int_{\mathbb{R}^3} g_2(\xi_2)g_3(\xi_3)g_4(\xi_4)g(\xi_2+\xi_3+\xi_4, \Omega(\xi_2,\xi_3,\xi_4))d\xi_2 d\xi_3 d\xi_4\\ &\lesssim 2^{-k_{max}/2}2^{k_{min}/2}\|g_2\|_{L^2}\|g_3\|_{L^2}\|g_4\|_{L^2}\|g\|_{L^2}.\end{aligned} \tag{4.11}$$

Indeed, by change of variables $\xi_2' = \xi_2, \xi_3' = \xi_3, \xi_4' = \xi_2+\xi_3+\xi_4$ and noting that in the area $|\xi_2'| \sim 2^{k_2}, |\xi_3'| \sim 2^{k_3}, |\xi_4'| \sim 2^{k_1}$,

$$\Big|\frac{\partial}{\partial_{\xi_2'}}\left[\Omega(\xi_2',\xi_3',\xi_4'-\xi_2'-\xi_3')\right]\Big| = |\omega'(\xi_2') - \omega'(\xi_4'-\xi_2'-\xi_3')| \sim 2^{k_3},$$

we get from Cauchy-Schwarz inequality that

$$\begin{aligned}
&\int_{\mathbb{R}^3} g_2(\xi_2)g_3(\xi_3)g_4(\xi_4)g(\xi_2+\xi_3+\xi_4,\Omega(\xi_2,\xi_3,\xi_4))d\xi_2 d\xi_3 d\xi_4\\
\lesssim\ & \int_{|\xi_2'|\sim 2^{k_2},|\xi_3'|\sim 2^{k_3},|\xi_4'|\sim 2^{k_1}} g_2(\xi_2')g_3(\xi_3')\\
&\cdot g_4(\xi_4'-\xi_2'-\xi_3')g(\xi_4',\Omega(\xi_2',\xi_3',\xi_4'-\xi_2'-\xi_3'))d\xi_2' d\xi_3' d\xi_4'\\
\lesssim\ & 2^{-k_3/2}\int_{|\xi_3'|\sim 2^{k_3},|\xi_4'|\sim 2^{k_1}} g_3(\xi_3')\|g_2(\xi_2')g_4(\xi_4'-\xi_2'-\xi_3')\|_{L^2_{\xi_2'}}\|g(\xi_4',\cdot)\|_{L^2_{\xi_2'}} d\xi_3' d\xi_4'\\
\lesssim\ & 2^{-k_{max}/2}2^{k_{min}/2}\|g_2\|_{L^2}\|g_3\|_{L^2}\|g_4\|_{L^2}\|g\|_{L^2}. \qquad (4.12)
\end{aligned}$$

We assume now that $j_2 = j_{max}$. This case is identical to the case $j_1 = j_{max}$. We note that we actually prove that if $k_2 \le k_3 - 5$ then

$$J \le C2^{(j_1+j_2+j_3+j_4)/2}2^{-j_{sub}/2}2^{-k_{max}/2}2^{k_{min}/2}\prod_{i=1}^{4}\|f_{k_i,j_i}\|_{L^2}. \qquad (4.13)$$

Therefore, we complete the proof for part (b).

For part (c), setting $f^\sharp_{k_i,j_i} = f_{k_i,j_i}(\xi,\tau-\omega(\xi))$, $i = 1,2,3$, then we get

$$\begin{aligned}
|J(f_{k_1,j_1},f_{k_2,j_2},f_{k_3,j_3},f_{k_4,j_4})| &= |\int f^\sharp_{k_1,j_1} * f^\sharp_{k_2,j_2} * f^\sharp_{k_3,j_3}\cdot f^\sharp_{k_4,j_4}|\\
&\lesssim \|f^\sharp_{k_1,j_1} * f^\sharp_{k_2,j_2} * f^\sharp_{k_3,j_3}\|_{L^2}\|f^\sharp_{k_4,j_4}|\|_{L^2}\\
&\lesssim \prod_{i=1}^{3}\|\mathcal{F}^{-1}(f^\sharp_{k_i,j_i})\|_{L^6}\|f_{k_4,j_4}|\|_{L^2}.
\end{aligned}$$

On the other hand, from

$$\begin{aligned}
\mathcal{F}^{-1}(f^\sharp_{k_1,j_1}) &= \int_{\mathbb{R}^2} f_{k_i,j_i}(\xi,\tau-\omega(\xi))e^{ix\xi}e^{it\tau}d\xi d\tau\\
&= \int_{\mathbb{R}^2} f_{k_i,j_i}(\xi,\tau)e^{ix\xi}e^{it\omega(\xi)}e^{it\tau}d\xi d\tau,
\end{aligned}$$

then it follows from Lemma 3.3 (a) that

$$\|\mathcal{F}^{-1}(f^\sharp_{k_1,j_1})\|_{L^6}\lesssim\int_{\mathbb{R}}\left\|\int_{\mathbb{R}} f_{k_i,j_i}(\xi,\tau)e^{ix\xi}e^{it\omega(\xi)}d\xi\right\|_{L^6}d\tau\lesssim 2^{j_i/2}\|f_{k_i,j_i}\|_{L^2}.$$

Thus part (c) follows form the symmetry.

For part (d), we only need to consider the worst cases $\xi_1\cdot\xi_2 < 0$ and $k_2 \le k_3-5$. Indeed in the other cases we get (4.6) from the fact $|\Omega(\xi_1,\xi_2,\xi_3)|\gtrsim 2^{k_2+k_3}$ which implies that $j_{max} \ge k_2+k_3-20$ by checking the support properties. Thus (d) follows from (b) and (c) in these cases. We assume now $\xi_1\cdot\xi_2 < 0$ and $k_2 \le k_3-5$. If $j_4 = j_{max}$, it suffices to prove that if g_i is L^2 nonnegative functions supported in I_{k_i}, $i = 1,2,3$, and g is a L^2 nonnegative function supported in $I_{k_4}\times\widetilde{I}_{j_4}$, then

$$\begin{aligned}
&\int_{\mathbb{R}^3\cap\{\xi_1\cdot\xi_2<0\}} g_1(\xi_1)g_2(\xi_2)g_3(\xi_3)g(\xi_1+\xi_2+\xi_3,\Omega(\xi_1,\xi_2,\xi_3))d\xi_1 d\xi_2 d\xi_3\\
\lesssim\ & 2^{j_4/2}2^{-k_3}\|g_1\|_{L^2}\|g_2\|_{L^2}\|g_3\|_{L^2}\|g\|_{L^2}. \qquad (4.14)
\end{aligned}$$

By localizing $|\xi_1+\xi_2|\sim 2^l$ for $l\in\mathbb{Z}$, we get that the right-hand side of (4.14) is bounded by

$$\sum_l\int_{\mathbb{R}^3}\chi_l(\xi_1+\xi_2)g_1(\xi_1)g_2(\xi_2)g_3(\xi_3)g(\xi_1+\xi_2+\xi_3,\Omega(\xi_1,\xi_2,\xi_3))d\xi_1d\xi_2d\xi_3. \tag{4.15}$$

From the support properties of the functions $g_i,\ g$ and the fact that in the integration area

$$|\Omega(\xi_1,\xi_2,\xi_3)|=(\xi_1+\xi_2)(\xi_1+\xi_3)\sim 2^{l+k_3},$$

We get that

$$j_{max}\geq l+k_3-20. \tag{4.16}$$

By changing variable of integration $\xi_1'=\xi_1+\xi_2$, $\xi_2'=\xi_2$, $\xi_3'=\xi_1+\xi_3$, we obtain that (4.15) is bounded by

$$\begin{aligned}&\sum_l\int_{|\xi_1'|\sim 2^l,|\xi_2'|\sim 2^{k_2},|\xi_3'|\sim 2^{k_3}}\chi_l(\xi_1')g_1(\xi_1'-\xi_2')g_2(\xi_2')g_3(\xi_2'+\xi_3'-\xi_1')\\&\quad g(\xi_2'+\xi_3',\Omega(\xi_1'-\xi_2',\xi_2',\xi_2'+\xi_3'-\xi_1'))d\xi_1'd\xi_2'd\xi_3'.\end{aligned} \tag{4.17}$$

Since in the integration area

$$\Big|\frac{\partial}{\partial_{\xi_1'}}[\Omega(\xi_1'-\xi_2',\xi_2',\xi_2'+\xi_3'-\xi_1')]\Big|=|\omega'(\xi_1'-\xi_2')-\omega'(\xi_2'+\xi_3'-\xi_1')|\sim 2^{k_3}, \tag{4.18}$$

then we get from (4.18) that (4.17) is bounded by

$$\begin{aligned}&\sum_l\int_{|\xi_1'|\sim 2^l}\chi_l(\xi_1')\|g_1\|_{L^2}\|g_3\|_{L^2}\\&\quad\|g_2(\xi_2')g(\xi_2'+\xi_3',\Omega(\xi_1'-\xi_2',\xi_2',\xi_2'+\xi_3'-\xi_1'))\|_{L^2_{\xi_2',\xi_3'}}d\xi_1'\\ \lesssim&\sum_l 2^{l/2}2^{-k_3/2}\|g_1\|_{L^2}\|g_2\|_{L^2}\|g_3\|_{L^2}\|g\|_{L^2}\\ \lesssim&\ 2^{j_{max}/2}2^{-k_3}\|g_1\|_{L^2}\|g_2\|_{L^2}\|g_3\|_{L^2}\|g\|_{L^2},\end{aligned} \tag{4.19}$$

where we used (4.16) in the last inequality.

From symmetry we know the case $j_3=j_{max}$ is identical to the case $j_4=j_{max}$, and the case $j_1=j_{max}$ is identical to the case $j_2=j_{max}$, thus it reduces to prove the case $j_2=j_{max}$. It suffices to prove that if g_i is L^2 nonnegative functions supported in I_{k_i}, $i=1,3,4$, and g is a L^2 nonnegative function supported in $I_{k_2}\times\widetilde{I}_{j_2}$, then

$$\begin{aligned}&\int_{\mathbb{R}^3\cap\{\xi_1\cdot\xi_2<0\}}g_1(\xi_1)g_3(\xi_3)g_4(\xi_4)g(\xi_1+\xi_3+\xi_4,\Omega(\xi_1,\xi_3,\xi_4))d\xi_1d\xi_3d\xi_4\\ \lesssim&\ 2^{j_2/2}2^{-k_3}\|g_1\|_{L^2}\|g_4\|_{L^2}\|g_3\|_{L^2}\|g\|_{L^2}.\end{aligned} \tag{4.20}$$

As the case $j_4=j_{max}$, we get that the right-hand side of (4.20) is bounded by

$$\sum_l\int_{\mathbb{R}^3}\chi_l(\xi_3+\xi_4)g_1(\xi_1)g_4(\xi_4)g_3(\xi_3)g(\xi_1+\xi_4+\xi_3,\Omega(\xi_1,\xi_3,\xi_4))d\xi_1d\xi_4d\xi_3. \tag{4.21}$$

From the support properties of the functions $g_i,\ g$ and the fact that in the integration area

$$|\Omega(\xi_1,\xi_2,\xi_3)|=|(\xi_1+\xi_4)(\xi_4+\xi_3)|\sim 2^{l+k_3},$$

We get that

$$j_{max} \geq l + k_3 - 20. \tag{4.22}$$

By changing variable of integration $\xi_1' = \xi_1 + \xi_3$, $\xi_3' = \xi_3 + \xi_4$, $\xi_4' = \xi_1 + \xi_3 + \xi_4$, we obtain that (4.21) is bounded by

$$\begin{aligned} &\sum_l \int_{|\xi_3'|\sim 2^l, |\xi_4'|\sim 2^{k_2}, |\xi_1'|\sim 2^{k_3}} \chi_l(\xi_3') g_1(\xi_4' - \xi_3') g_3(\xi_1' + \xi_3' - \xi_4') g_4(\xi_4' - \xi_1') \\ &\quad g(\xi_4', \Omega(\xi_4' - \xi_3', \xi_1' + \xi_3' - \xi_4', \xi_4' - \xi_1')) d\xi_1' d\xi_3' d\xi_4'. \end{aligned} \tag{4.23}$$

Since in the integration area,

$$\begin{aligned} & \left|\frac{\partial}{\partial_{\xi_3'}}[\Omega(\xi_4' - \xi_3', \xi_1' + \xi_3' - \xi_4', \xi_4' - \xi_1')]\right| \\ = \quad & |-\omega'(\xi_4' - \xi_3') + \omega'(\xi_1' + \xi_3' - \xi_4')| \sim 2^{k_3}, \end{aligned} \tag{4.24}$$

then we get from (4.24) that (4.23) is bounded by

$$\begin{aligned} & \sum_l \int_{|\xi_3'|\sim 2^l} \chi_l(\xi_3') \|g_1\|_{L^2} \|g_3\|_{L^2} \\ & \quad \|g_4(\xi_4' - \xi_1') g(\xi_4', \Omega(\xi_4' - \xi_3', \xi_1' + \xi_3' - \xi_4', \xi_4' - \xi_1'))\|_{L^2_{\xi_1',\xi_4'}} d\xi_3' \\ \lesssim \quad & \sum_l 2^{l/2} 2^{-k_3/2} \|g_1\|_{L^2} \|g_3\|_{L^2} \|g_4\|_{L^2} \|g\|_{L^2} \\ \lesssim \quad & 2^{j_{max}/2} 2^{-k_3} \|g_1\|_{L^2} \|g_2\|_{L^2} \|g_3\|_{L^2} \|g\|_{L^2}, \end{aligned} \tag{4.25}$$

where we used (4.22) in the last inequality. Therefore, we complete the proof for part (d). ∎

We restate now Lemma 4.1 in a form that is suitable for the trilinear estimates in the next sections.

Corollary 4.2. *Assume $k_1, k_2, k_3, k_4 \in \mathbb{Z}$, $j_1, j_2, j_3, j_4 \in \mathbb{Z}_+$, and $f_{k_i,j_i} \in L^2(\mathbb{R}\times\mathbb{R})$ are functions supported in $\dot{D}_{k_i,j_i}$, $i = 1, 2$.*

(a) For any $k_1, k_2, k_3, k_4 \in \mathbb{Z}$ and $j_1, j_2, j_3, j_4 \in \mathbb{Z}_+$,

$$\begin{aligned} & \|1_{\dot{D}_{k_4,j_4}}(\xi,\tau)(f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3})\|_{L^2} \\ \leq \quad & C 2^{(k_{min}+k_{thd})/2} 2^{(j_{min}+j_{thd})/2} \prod_{i=1}^{3} \|f_{k_i,j_i}\|_{L^2}. \end{aligned} \tag{4.26}$$

(b) For any $k_1, k_2, k_3, k_4 \in \mathbb{Z}$ with $k_{thd} \leq k_{max} - 5$, and $j_1, j_2, j_3, j_4 \in \mathbb{Z}_+$. If for some $i \in \{1, 2, 3, 4\}$ such that $(k_i, j_i) = (k_{thd}, j_{max})$, then

$$\begin{aligned} & \|1_{\dot{D}_{k_4,j_4}}(\xi,\tau)(f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3})\|_{L^2} \\ \leq \quad & C 2^{(-k_{max}+k_{thd})/2} 2^{(j_1+j_2+j_3+j_4)/2} 2^{-j_{max}/2} \prod_{i=1}^{3} \|f_{k_i,j_i}\|_{L^2}, \end{aligned} \tag{4.27}$$

else we have

$$\begin{aligned} & \|1_{\dot{D}_{k_4,j_4}}(\xi,\tau)(f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3})\|_{L^2} \\ \leq \quad & C 2^{(-k_{max}+k_{min})/2} 2^{(j_1+j_2+j_3+j_4)/2} 2^{-j_{max}/2} \prod_{i=1}^{3} \|f_{k_i,j_i}\|_{L^2}. \end{aligned} \tag{4.28}$$

(c) For any $k_1, k_2, k_3, k_4 \in \mathbb{Z}$ and $j_1, j_2, j_3, j_4 \in \mathbb{Z}_+$,

$$\begin{aligned} & \|1_{\dot{D}_{k_4,j_4}}(\xi,\tau)(f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3})\|_{L^2} \\ \leq\ & C 2^{(j_1+j_2+j_3+j_4)/2} 2^{-j_{max}/2} \prod_{i=1}^{3} \|f_{k_i,j_i}\|_{L^2}. \end{aligned} \tag{4.29}$$

(d) For any $k_1, k_2, k_3, k_4 \in \mathbb{Z}$ with $k_{min} \leq k_{max} - 10$, and $j_1, j_2, j_3, j_4 \in \mathbb{Z}_+$,

$$\begin{aligned} & \|1_{\dot{D}_{k_4,j_4}}(\xi,\tau)(f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3})\|_{L^2} \\ \leq\ & C 2^{(j_1+j_2+j_3+j_4)/2} 2^{-k_{max}/2} \prod_{i=1}^{3} \|f_{k_i,j_i}\|_{L^2}. \end{aligned} \tag{4.30}$$

Proof. Clearly, we have

$$\begin{aligned} & \|1_{\dot{D}_{k_4,j_4}}(\xi,\tau)(f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3})(\xi,\tau)\|_{L^2} \\ =\ & \sup_{\|f\|_{L^2}=1} \left| \int_{D_{k_4,j_4}} f \cdot f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3} d\xi d\tau \right|. \end{aligned} \tag{4.31}$$

Let $f_{k_3,j_3} = 1_{D_{k_4,j_4}} \cdot f$, and then $f^\sharp_{k_i,j_i}(\xi,\mu) = f_{k_i,j_i}(\xi, \mu + \omega(\xi))$, $i = 1,2,3,4$. The functions $f^\sharp_{k_i,j_i}$ are supported in $I_{k_i} \times \cup_{|m|\leq 3} \widetilde{I}_{j_i+m}$, $\|f^\sharp_{k_i,j_i}\|_{L^2} = \|f_{k_i,j_i}\|_{L^2}$. Using simple changes of variables, we get

$$\int_{D_{k_4,j_4}} f \cdot f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3} d\xi d\tau = J(f^\sharp_{k_1,j_1}, f^\sharp_{k_2,j_2}, f^\sharp_{k_3,j_3}, f^\sharp_{k_4,j_4}).$$

Then Corollary 4.2 follows from Lemma 4.1. ∎

Remark 4.3. From the proof, we see that Lemma 4.1 and Corollary 4.2 also hold if $k_1, k_2, k_3, k_4 \in \mathbb{Z}_+$ and with I_{k_i} replaced by $\widetilde{I}_{k_i}$, $\dot{D}_{k_i,j_i}$ replaced by $\cup_{l=0}^{j_i} D_{k_i,l}$. The methods can be used to deal with a general dispersion relation $\omega(\xi)$.

5. Trilinear Estimates

In this section we devote to prove some dyadic trilinear estimates, using the symmetric estimates obtained in the last section. We divide it into several cases. The first case is *low* $\times$ *high* $\to$ *high* interactions.

Proposition 5.1. *Assume $k_3 \geq 110$, $|k_4 - k_3| \leq 5$, $0 \leq k_1, k_2 \leq k_3 - 10$, $|k_1 - k_2| \leq 10$, and $f_{k_i} \in Z_{k_i}$ with $\mathcal{F}^{-1}(f_{k_i})$ compactly supported (in time) in J_0, $|J_0| \lesssim 1$, $i = 1,2,3$. Then*

$$2^{k_4} \|\eta_{k_4}(\xi)(\tau - \omega(\xi) + i)^{-1} f_{k_1} * f_{k_2} * f_{k_3}\|_{Z_{k_4}} \lesssim 2^{(k_1+k_2)/2} \prod_{i=1}^{3} \|f_{k_i}\|_{Z_{k_i}}. \tag{5.1}$$

Proof. We first divide it into three parts, according to the modulation.

$$\begin{aligned} & 2^{k_4} \|\chi_{k_4}(\xi)(\tau - \omega(\xi) + i)^{-1} f_{k_1} * f_{k_2} * f_{k_3}\|_{Z_{k_4}} \\ \leq\ & 2^{k_4} \|\chi_{k_4}(\xi)\eta_{\leq k_4 - 1}(\tau - \omega(\xi))(\tau - \omega(\xi) + i)^{-1} f_{k_1} * f_{k_2} * f_{k_3}\|_{Z_{k_4}} \\ & + 2^{k_4} \|\chi_{k_4}(\xi)\eta_{[k_4, 2k_4]}(\tau - \omega(\xi))(\tau - \omega(\xi) + i)^{-1} f_{k_1} * f_{k_2} * f_{k_3}\|_{Z_{k_4}} \\ & + 2^{k_4} \|\chi_{k_4}(\xi)\eta_{\geq 2k_4+1}(\tau - \omega(\xi))(\tau - \omega(\xi) + i)^{-1} f_{k_1} * f_{k_2} * f_{k_3}\|_{Z_{k_4}} \\ =\ & I + II + III. \end{aligned}$$

We consider first the contribution of I. Using Y_k norm, then we get from Lemma 3.1 (a), (c) and Lemma 3.4 that

$$\begin{aligned}
I &\leq 2^{k_4}\|\chi_{k_4}(\xi)\eta_{\leq k_4-1}(\tau-\omega(\xi))(\tau-\omega(\xi)+i)^{-1}f_{k_1}*f_{k_2}*f_{k_3}\|_{Y_{k_4}}\\
&\lesssim 2^{k_4/2}\|\mathcal{F}^{-1}[f_{k_1}*f_{k_2}*f_{k_3}]\|_{L_x^1L_t^2}\\
&\lesssim 2^{k_4/2}\|\mathcal{F}^{-1}(f_{k_3})\|_{L_x^\infty L_t^2}\|\mathcal{F}^{-1}(f_{k_2})\|_{L_x^2L_{t\in I_0}^\infty}\|\mathcal{F}^{-1}(f_{k_1})\|_{L_x^2L_{t\in I_0}^\infty}\\
&\lesssim 2^{(k_1+k_2)/2}\prod_{i=1}^3\|f_{k_i}\|_{Z_{k_i}},
\end{aligned}$$

which is (5.1) as desired.

For the contribution of II, we use X_k norm. Then we get from Lemma 3.4 that

$$\begin{aligned}
II &\leq 2^{k_4}\|\chi_{k_4}(\xi)\eta_{[k_4,2k_4]}(\tau-\omega(\xi))(\tau-\omega(\xi)+i)^{-1}f_{k_1}*f_{k_2}*f_{k_3}\|_{Z_{k_4}}\\
&\leq \sum_{k_4\leq j\leq 2k_4}2^{k_4}2^{-j/2}\|1_{D_{k_4,j}}(\xi,\tau)f_{k_1}*f_{k_2}*f_{k_3}\|_{L^2}\\
&\leq \sum_{k_4\leq j\leq 2k_4}2^{k_4}2^{-j/2}\|\mathcal{F}^{-1}(f_{k_3})\|_{L_x^\infty L_t^2}\|\mathcal{F}^{-1}(f_{k_1})\|_{L_x^4L_t^\infty}\|\mathcal{F}^{-1}(f_{k_2})\|_{L_x^4L_t^\infty}\\
&\lesssim 2^{(k_1+k_2)/4}\prod_{i=1}^3\|f_{k_i}\|_{Z_{k_i}}, \qquad (5.2)
\end{aligned}$$

which is acceptable.

Finally we consider the contribution of III. For $j_i\geq 0$, $i=1,2,3$, let $f_{k_i,j_i}(\xi,\tau)=f_{k_i}(\xi,\tau)\eta_{j_i}(\tau-\omega(\xi))$. Using X_k norm, we get

$$III\leq \sum_{j_4\geq 2k_4+1}\sum_{j_1,j_2,j_3\geq 0}\|1_{D_{k_4,j_4}}(\xi,\tau)f_{k_1,j_1}*f_{k_2,j_2}*f_{k_3,j_3}\|_{L^2}. \qquad (5.3)$$

Since in the area $\{|\xi_i|\in\widetilde{I}_{k_i}, i=1,2,3\}$, we have $|\Omega(\xi_1,\xi_2,\xi_3)|\ll 2^{2k_4}$. By checking the support properties of f_{k_i,j_i}, we get $|j_{max}-j_{sub}|\leq 5$. We consider only the worst case $|j_4-j_3|\leq 5$, since the other cases are better. It follows from Corollary 4.2 and Lemma 3.1 (b) that

$$\begin{aligned}
III &\lesssim \sum_{j_3\geq 2k_4+1}\sum_{j_1,j_2\geq 0}2^{(j_1+j_2)/2}2^{(k_1+k_2)/2}\|f_{k_1,j_1}\|_{L^2}\|f_{k_2,j_2}\|_{L^2}\|f_{k_3,j_3}\|_{L^2}\\
&\lesssim \sum_{j_3\geq 2k_4+1}2^{j_3/4}2^{k_3-j_3}2^{(k_1+k_2)/2}2^{j_3-k_3}\|f_{k_1}\|_{Z_{k_1}}\|f_{k_2}\|_{Z_{k_2}}\|f_{k_3,j_3}\|_{L^2}\\
&\lesssim \sum_{j_3\geq 2k_4+1}2^{k_3-\frac{3}{4}j_3}2^{(k_1+k_2)/2}\|f_{k_1}\|_{Z_{k_1}}\|f_{k_2}\|_{Z_{k_2}}\|f_{k_3}\|_{Z_{k_3}}\\
&\lesssim 2^{(k_1+k_2)/4}\|f_{k_1}\|_{Z_{k_1}}\|f_{k_2}\|_{Z_{k_2}}\|f_{k_3}\|_{Z_{k_3}}. \qquad (5.4)
\end{aligned}$$

Therefore, we complete the proof of the proposition. ■

This proposition suffices to control $high\times low$ interaction in the case that the two low frequences are comparable. However, for the case that the two low frequences are not comparable, we will need an improvement.

Proposition 5.2. *Assume* $k_3\geq 110$, $|k_4-k_3|\leq 5$, $0\leq k_1,k_2\leq k_3-10$, $k_2\geq 10$, $k_1\leq k_2-5$ *and* $f_{k_i}\in Z_{k_i}$ *with* $\mathcal{F}^{-1}(f_{k_i})$ *compactly supported (in time) in* J_0,

$|J_0|\lesssim 1$, $i=1,2,3$. *Then*

$$2^{k_4}\|\chi_{k_4}(\xi)(\tau-\omega(\xi)+i)^{-1}f_{k_1}*f_{k_2}*f_{k_3}\|_{Z_{k_4}}\leq C2^{(k_1+k_2)/4}\prod_{i=1}^{3}\|f_{k_i}\|_{Z_{k_i}}. \tag{5.5}$$

Proof. We first observe that in this case we have

$$|\Omega(\xi_1,\xi_2,\xi_3)|\sim 2^{k_3+k_2}, \tag{5.6}$$

which follows from the fact that $\xi_1+\xi_2+\xi_3$ has the same sign as ξ_3, $k_1\leq k_2-10$ and in the area $\{\xi_i\in \widetilde{I}_{k_i}, i=1,2,3\}$

$$|\omega(\xi_3)-\omega(\xi_1+\xi_2+\xi_3)|\sim 2^{k_3+k_2}. \tag{5.7}$$

Dividing it into three parts as before, we obtain

$$\begin{aligned}
&2^{k_4}\|\chi_{k_4}(\xi)(\tau-\omega(\xi)+i)^{-1}f_{k_1}*f_{k_2}*f_{k_3}\|_{Z_{k_4}}\\
\leq\ &2^{k_4}\|\chi_{k_4}(\xi)\eta_{\leq k_4-1}(\tau-\omega(\xi))(\tau-\omega(\xi)+i)^{-1}f_{k_1}*f_{k_2}*f_{k_3}\|_{Z_{k_4}}\\
&+2^{k_4}\|\chi_{k_4}(\xi)\eta_{[k_4,2k_4]}(\tau-\omega(\xi))(\tau-\omega(\xi)+i)^{-1}f_{k_1}*f_{k_2}*f_{k_3}\|_{Z_{k_4}}\\
&+2^{k_4}\|\chi_{k_4}(\xi)\eta_{\geq 2k_4+1}(\tau-\omega(\xi))(\tau-\omega(\xi)+i)^{-1}f_{k_1}*f_{k_2}*f_{k_3}\|_{Z_{k_4}}\\
=\ &I+II+III.
\end{aligned}$$

For the last two terms II, III, we can use the same argument as for II, III in the proof of Proposition 5.1. We consider now the first term I.

$$\begin{aligned}
I\ \leq\ &2^{k_4}\|\chi_{k_4}(\xi)\eta_{\leq k_4-1}(\tau-\omega(\xi))(\tau-\omega(\xi)+i)^{-1}f_{k_1}*f_{k_2}*f_{k_3}^h\|_{Z_{k_4}}\\
&+2^{k_4}\|\chi_{k_4}(\xi)\eta_{\leq k_4-1}(\tau-\omega(\xi))(\tau-\omega(\xi)+i)^{-1}f_{k_1}*f_{k_2}*f_{k_3}^l\|_{Z_{k_4}}\\
=\ &I_1+I_2,
\end{aligned}$$

where

$$f_{k_3}^h=f_{k_3}(\xi,\tau)\eta_{\geq k+k_2-10}(\tau-\omega(\xi)),\quad f_{k_3}^l=f_{k_3}(\xi,\tau)\eta_{\leq k+k_2-9}(\tau-\omega(\xi)).$$

For the contribution of I_1, we observe first that from the support of $f_{k_3}^h$ and the definition of Y_k, one easily get that

$$\|f_{k_3}^h\|_{X_{k_3}}\lesssim\|f_{k_3}\|_{Z_{k_3}}. \tag{5.8}$$

Thus from the definition of Y_k, and from Hölder's inequality, Lemma 3.1 (a), (c) and Lemma 3.4, we get

$$\begin{aligned}
I_1\ \lesssim\ &2^{k_4}\|\chi_{k_4}(\xi)\eta_{\leq k_4-1}(\tau-\omega(\xi))(\tau-\omega(\xi)+i)^{-1}f_{k_1}*f_{k_2}*f_{k_3}^h\|_{Y_{k_4}}\\
\lesssim\ &2^{k_4/2}\|\mathcal{F}^{-1}[f_{k_1}*f_{k_2}*f_{k_3}^h]\|_{L_x^1L_t^2}\\
\lesssim\ &2^{k_4/2}\|\mathcal{F}^{-1}(f_{k_3}^h)\|_{L_x^2L_t^2}\|\mathcal{F}^{-1}(f_{k_1})\|_{L_x^4L_t^\infty}\|\mathcal{F}^{-1}(f_{k_2})\|_{L_x^4L_t^\infty}\\
\lesssim\ &2^{k_4/2}2^{(k_1+k_2)/4}\|f_{k_3}^h\|_{L^2}\|f_{k_1}\|_{Z_{k_1}}\|f_{k_2}\|_{Z_{k_2}}.
\end{aligned}$$

Then from the fact that

$$\begin{aligned}
2^{k_4/2}\|f_{k_3}^h\|_{L^2}\ \lesssim\ &\sum_{j\geq k_4+k_2-10}2^{k_4/2}\|f_{k_3}^h\eta_j(\tau-\omega(\xi))\|_{L^2}\\
\lesssim\ &\|f_{k_3}^h\|_{X_{k_3}}\lesssim\|f_{k_3}\|_{Z_{k_3}}
\end{aligned} \tag{5.9}$$

we conclude the proof for I_1.

We consider now the contribution of I_2. Let $f_{k_i,j_i}(\xi,\tau)=f_{k_i}(\xi,\tau)\eta_{j_i}(\tau-\omega(\xi))$, $j_i\geq 0$, $i=1,2,3$. Using X_k norm, we get

$$I_2\leq \sum_{j_4\leq k_4-1}\sum_{j_3\leq k_4+k_2-9}\sum_{j_1,j_2\geq 0}2^{k_4}2^{-j_4/2}\|1_{D_{k_4,j_4}}(\xi,\tau)f_{k_1,j_1}*f_{k_2,j_2}*f_{k_3,j_3}\|_{L^2}.$$

By checking the support properties of the functions f_{k_i,j_i} $(i=1,2,3)$ and from (5.6), we easily get that $1_{D_{k_4,j_4}}(\xi,\tau)f_{k_1,j_1}*f_{k_2,j_2}*f_{k_3,j_3}\equiv 0$ unless

$$\begin{cases} j_1,j_2\geq k_3+k_2-10, |j_1-j_2|\leq 5;\ \text{or} \\ |j_1-k_3-k_2|\leq 5, j_2\leq k_3+k_2-10;\ \text{or}\ |j_2-k_3-k_2|\leq 5, j_1\leq k_3+k_2-10. \end{cases}$$

Case 1. $j_1,j_2\geq k_3+k_2-10, |j_1-j_2|\leq 5$.

It follows from Corollary 4.2 (b) and Lemma 3.1 (b) that

$$\begin{aligned} I_2 &\lesssim \sum_{j_3,j_4\leq 2k_4}\sum_{j_1,j_2\geq 0}2^{k_4}2^{-j_4/2}2^{(j_1+j_2+j_3+j_4)/2}2^{-j_1/2}2^{-k_3/2}2^{k_1/2}\prod_{i=1}^{3}\|f_{k_i,j_i}\|_2 \\ &\lesssim \sum_{j_1,j_2\geq 0}k_4^2 2^{j_2/2}2^{k_3/2}2^{k_1/2}\prod_{i=1}^{2}\|f_{k_i,j_i}\|_2\|f_{k_3}\|_{Z_{k_3}} \\ &\lesssim \sum_{j_1,j_2\geq 0}k_4^2 2^{j_2/2}2^{k_3/2}2^{k_1/2}2^{k_1+k_2-j_1-j_2}\prod_{i=1}^{2}(2^{j_i-k_i}\|f_{k_i,j_i}\|_2)\|f_{k_3}\|_{Z_{k_3}} \\ &\lesssim 2^{(k_1+k_2)/4}\prod_{i=1}^{3}\|f_{k_i}\|_{Z_{k_i}}, \end{aligned} \tag{5.10}$$

which is acceptable.

Case 2. $|j_1-k_3-k_2|\leq 5, j_2\leq k_3+k_2-10$; or $|j_2-k_3-k_2|\leq 5, j_1\leq k_3+k_2-10$.

We consider only the worse case $|j_2-k_3-k_2|\leq 5,\ j_1\leq k_3+k_2-10$. It follows from Corollary (b) that

$$\begin{aligned} I_2 &\lesssim \sum_{j_1,j_3,j_4\leq 2k_4}\sum_{j_2\geq 0}2^{k_4}2^{(j_1+j_3)/2}2^{-k_3/2}2^{k_2/2}\prod_{i=1}^{3}\|f_{k_i,j_i}\|_2 \\ &\lesssim k_4^3 2^{k_4}2^{k_2-j_2}2^{-k_3/2}2^{k_2/2}\prod_{i=1}^{3}\|f_{k_i}\|_{Z_{k_i}} \\ &\lesssim 2^{(k_1+k_2)/4}\prod_{i=1}^{3}\|f_{k_i}\|_{Z_{k_i}}. \end{aligned}$$

Therefore, we complete the proof of the proposition. ■

Proposition 5.3. *Assume $k_3\geq 110$, $|k_4-k_3|\leq 5$, $k_3-10\leq k_2\leq k_3$, $0\leq k_1\leq k_2-10$ and $f_{k_i}\in Z_{k_i}$ with $\mathcal{F}^{-1}(f_{k_i})$ compactly supported (in time) in J_0, $|J_0|\lesssim 1$, $i=1,2,3$. Then*

$$2^{k_4}\|\chi_{k_4}(\xi)(\tau-\omega(\xi)+i)^{-1}f_{k_1}*f_{k_2}*f_{k_3}\|_{Z_{k_4}}\lesssim 2^{(k_1+k_2)/4}\prod_{i=1}^{3}\|f_{k_i}\|_{Z_{k_i}}. \tag{5.11}$$

Proof. We first observe that this case corresponds to an integration in the area $\{|\xi_i|\in I_{k_i},\ i=1,2,3\}\cap\{|\xi_1+\xi_2+\xi_3|\in I_{k_4}\}$, where we have

$$|\Omega(\xi_1,\xi_2,\xi_3)|\sim 2^{2k_3}. \tag{5.12}$$

Let $f_{k_i,j_i}(\xi,\tau)=f_{k_i}(\xi,\tau)\eta_{j_i}(\tau-\omega(\xi))$, $j_i\geq 0$ and $i=1,2,3$. Using X_k norm, we get

$$\begin{aligned}&2^{k_4}\|\chi_{k_4}(\xi)(\tau-\omega(\xi)+i)^{-1}f_{k_1}*f_{k_2}*f_{k_3}\|_{Z_{k_4}}\\ \lesssim&\sum_{j_1,j_2,j_3,j_4\geq 0}2^{k_4}2^{-j_4/2}(1+2^{(j_4-2k_4)/2})\|1_{D_{k_4,j_4}}(\xi,\tau)f_{k_1,j_1}*f_{k_2,j_2}*f_{k_3,j_3}\|_{L^2}.\end{aligned}$$

From the support properties of the functions f_{k_i,j_i}, $i=1,2,3$, it is easy to see that $1_{D_{k_4,j_4}}(\xi,\tau)f_{k_1,j_1}*f_{k_2,j_2}*f_{k_3,j_3}\equiv 0$ unless

$$\begin{cases} j_{max},j_{sub}\geq 2k_3-10, |j_{max}-j_{sub}|\leq 5;\ \text{or}\\ |j_{max}-2k_3|\leq 5, j_{sub}\leq 2k_3-10.\end{cases}$$

Case 1. $j_{max},j_{sub}\geq 2k_3-10, |j_{max}-j_{sub}|\leq 5$.

It follows from Corollary 4.2 (a) that the right-hand side of (5.13) is bounded by

$$\begin{aligned}&\sum_{j_1,j_2,j_3,j_4\geq 0}2^{k_4}2^{(j_1+j_2+j_3)/2}(1+2^{(j_4-2k_4)/2})\\ &\qquad 2^{-(j_{sub}+j_{max})/2}2^{(k_1+k_2)/2}\prod_{i=1}^{3}\|f_{k_i,j_i}\|_2.\end{aligned}\tag{5.13}$$

It suffices to consider the worst case $j_3,j_4=j_{max},j_{sub}$. We get from Lemma 3.1 (b) that (5.13) is bounded by

$$\sum_{j_3\geq 2k_3-10}2^{k_4}2^{-\frac{3}{4}j_3}2^{(k_1+k_2)/2}\prod_{i=1}^{3}\|f_{k_i}\|_{Z_{k_i}}\lesssim 2^{(k_1+k_2)/4}\prod_{i=1}^{3}\|f_{k_i}\|_{Z_{k_i}}.\tag{5.14}$$

Case 2. $|j_{max}-2k_3|\leq 5, j_{sub}\leq 2k_3-10$.

From Corollary 4.2 (c), we get that the right-hand side of (5.13) is bounded by

$$\begin{aligned}&\sum_{j_1,j_2,j_3,j_4\geq 0}2^{k_4}2^{(j_1+j_2+j_3)/2}2^{-j_{max}/2}\prod_{i=1}^{3}\|f_{k_i,j_i}\|_2\\ \lesssim&2^{(k_1+k_2)/4}\prod_{i=1}^{3}\|f_{k_i}\|_{Z_{k_i}},\end{aligned}\tag{5.15}$$

where we used Lemma 3.1 (b). Thus, we complete the proof of the proposition. ∎

Proposition 5.4. *Assume $k_3\geq 110$, $|k_4-k_3|\leq 5$, $k_3-30\leq k_1,\ k_2\leq k_3$, and $f_{k_i}\in Z_{k_i}$ with $\mathcal{F}^{-1}(f_{k_i})$ compactly supported (in time) in J_0, $|J_0|\lesssim 1$, $i=1,2,3$. Then*

$$2^{k_4}\|\chi_{k_4}(\xi)(\tau-\omega(\xi)+i)^{-1}f_{k_1}*f_{k_2}*f_{k_3}\|_{Z_{k_4}}\leq C2^{k_3}\prod_{i=1}^{3}\|f_{k_i}\|_{Z_{k_i}}.\tag{5.16}$$

Proof. First we divide it into two parts.

$$\begin{aligned}&2^{k_4}\|\chi_{k_4}(\xi)(\tau-\omega(\xi)+i)^{-1}f_{k_1}*f_{k_2}*f_{k_3}\|_{Z_{k_4}}\\ \lesssim&2^{k_4}\|\chi_{k_4}(\xi)\eta_{\leq 2k_4+20}(\tau-\omega(\xi))(\tau-\omega(\xi)+i)^{-1}f_{k_1}*f_{k_2}*f_{k_3}\|_{Z_{k_4}}\\ &+2^{k_4}\|\chi_{k_4}(\xi)\eta_{\geq 2k_4+21}(\tau-\omega(\xi))(\tau-\omega(\xi)+i)^{-1}f_{k_1}*f_{k_2}*f_{k_3}\|_{Z_{k_4}}\\ =&I+II.\end{aligned}$$

We consider first the contribution of the first term I. Using the X_k norm and Lemma 3.4, then we get

$$\begin{aligned} I &\lesssim 2^{k_4}\sum_{j_4\geq 0}^{2k_4+20} 2^{-j_4/2}\|1_{D_{k_4,j_4}}(\xi,\tau)f_{k_1}*f_{k_2}*f_{k_3}\|_{L^2} \\ &\lesssim 2^{k_4}\prod_{i=1}^{3}\|\mathcal{F}^{-1}(f_{k_i})\|_{L^6}\lesssim 2^{k_4}\prod_{i=1}^{3}\|f_{k_i}\|_{Z_{k_i}}. \end{aligned} \tag{5.17}$$

We consider now the contribution of the second term II. Let $f_{k_i,j_i}(\xi,\tau) = f_{k_i}(\xi,\tau)\eta_{j_i}(\tau-\omega(\xi))$, $j_i \geq 0$, $i = 1,2,3$. Using the X_k norm, we get

$$II \lesssim \sum_{j_4\geq 2k_4+20}\ \sum_{j_1,j_2,j_3\geq 0}\|1_{D_{k_4,j_4}}(\xi,\tau)f_{k_1,j_1}*f_{k_2,j_2}*f_{k_3,j_3}\|_{L^2}. \tag{5.18}$$

Since in the area $\{|\xi_i| \in I_{k_i},\ i = 1,2,3\}$ we have $|\Omega(\xi_1,\xi_2,\xi_3)|\lesssim 2^{2k_3}$, by checking the support properties of the functions f_{k_i,j_i}, $i = 1,2,3$, we get $|j_{max} - j_{sub}| \leq 5$ and $j_{sub} \geq 2k_3 + 10$. From symmetry, we assume $j_3, j_4 = j_{max}, j_{sub}$, then we get

$$\begin{aligned} II &\lesssim \sum_{j_4\geq 2k_4+20}\ \sum_{j_1,j_2,j_3\geq 0} 2^{(j_1+j_2)/2}2^{k_3}2^{k_3-j_3}2^{j_3-k_3}\prod_{i=1}^{3}\|f_{k_i,j_i}\|_2 \\ &\lesssim \sum_{j_3\geq 2k_4+20} 2^{2k_3}2^{-j_3/2}\prod_{i=1}^{3}\|f_{k_i}\|_{Z_{k_i}}\lesssim 2^{k_4}\prod_{i=1}^{3}\|f_{k_i}\|_{Z_{k_i}}. \end{aligned} \tag{5.19}$$

Therefore we complete the proof of the proposition. ∎

We consider now the case which corresponds to *high* $\times$ *high* interactions. This case is better than *high* $\times$ *low* interaction case.

Proposition 5.5. *Assume $k_1 \geq 110$, $|k_1 - k_2| \leq 5$, $0 \leq k_3 \leq k_1 + 10$, $0 \leq k_4 \leq k_1 - 10$, and $f_{k_i} \in Z_{k_i}$ with $\mathcal{F}^{-1}(f_{k_i})$ compactly supported (in time) in J_0, $|J_0|\lesssim 1$, $i = 1,2,3$. Then*

$$2^{k_4}\|\eta_{k_4}(\xi)(\tau-\omega(\xi)+i)^{-1}f_{k_1}*f_{k_2}*f_{k_3}\|_{Z_{k_4}} \leq Ck_1^4\prod_{i=1}^{3}\|f_{k_i}\|_{Z_{k_i}} \tag{5.20}$$

Proof. We divide the argument into two cases. Let $f_{k_i,j_i}(\xi,\tau) = f_{k_i}(\xi,\tau)\eta_{j_i}(\tau-\omega(\xi))$, $j_i \geq 0$, $i = 1,2,3$. Using X_k norm, then we get

$$\begin{aligned} &2^{k_4}\|\chi_{k_4}(\xi)(\tau-\omega(\xi)+i)^{-1}f_{k_1}*f_{k_2}*f_{k_3}\|_{Z_{k_4}} \\ \lesssim &\sum_{j_i\geq 0} 2^{k_4}2^{-j_4/2}(1+2^{(j_4-2k_4)/2})\|1_{D_{k_4,j_4}}(\xi,\tau)f_{k_1,j_1}*f_{k_2,j_2}*f_{k_3,j_3}\|_{L^2}. \end{aligned} \tag{5.21}$$

Case 1. $\max(j_1,j_2,j_3,j_4) \leq 2k_1 + 20$.

It follows from Corollary 4.2 (d) that the right-hand side of (5.21) is bounded by

$$\sum_{j_i\geq 0} 2^{k_4}(1+2^{(j_4-2k_4)/2})2^{(j_1+j_2+j_3)/2}2^{-k_1}\prod_{i=1}^{3}\|f_{k_i,j_i}\|_{L^2}\lesssim k_1^4\prod_{i=1}^{3}\|f_{k_i}\|_{Z_{k_i}}, \tag{5.22}$$

where we used Lemma 3.1 (b).

Case 2. $\max(j_1,j_2,j_3,j_4) \geq 2k_1 + 20$.

By checking the support properties, we get $|j_{max}-j_{sub}|\le 5$. We consider only the worst case $j_1, j_4 = j_{max}, j_{sub}$. It follows from Corollary 4.2 (a) and Lemma 3.1 (b) that the right side of (5.21) is bounded by

$$\sum_{j_i\ge 0} 2^{k_4}2^{-j_4/2}(1+2^{(j_4-2k_4)/2})2^{(j_2+j_3)/2}2^{k_4}\prod_{i=1}^{3}\|f_{k_i,j_i}\|_{L^2}\lesssim k_1^4\prod_{i=1}^{3}\|f_{k_i}\|_{Z_{k_i}}. \quad (5.23)$$

Therefore, we complete the proof of the proposition. ∎

The next proposition is used to control $low\times low$ interactions. This interaction is easy to control.

Proposition 5.6. *Assume* $0\le k_1,k_2,k_3,k_4\le 120$, *and* $f_{k_i}\in Z_{k_i}$, $i=1,2,3$. *Then*

$$2^{k_4}\|\eta_{k_4}(\xi)(\tau-\omega(\xi)+i)^{-1}f_{k_1}*f_{k_2}*f_{k_3}\|_{Z_{k_4}}\le C\prod_{i=1}^{3}\|f_{k_i}\|_{Z_{k_i}}. \quad (5.24)$$

Proof. Let $f_{k_i,j_i}(\xi,\tau)=f_{k_i}(\xi,\tau)\eta_{j_i}(\tau-\omega(\xi))$, $j_i\ge 0$, $i=1,2,3$. Using X_k norm, Corollary 4.2 (a) and Lemma 3.1 (b), then we get

$$\begin{aligned}
&2^{k_4}\|\chi_{k_4}(\xi)(\tau-\omega(\xi)+i)^{-1}f_{k_1}*f_{k_2}*f_{k_3}\|_{Z_{k_4}}\\
\lesssim\ &\sum_{j_1,j_2,j_3,j_4\ge 0}2^{(j_{min}+j_{thd})/2}2^{(k_{min}+k_{thd})/2}\prod_{i=1}^{3}\|f_{k_i,j_i}\|_{L^2}\\
\lesssim\ &2^{(k_{min}+k_{thd})/2}\prod_{i=1}^{3}\|f_{k_i}\|_{Z_{k_i}},
\end{aligned}$$

since for the case $j_{max}\ge 200$ we have $|j_{max}-j_{sub}|\le 5$ by checking the support properties of the functions f_{k_i,j_i}, $i=1,2,3$. ∎

Finally we present two counterexamples. The first one shows why we use a l^1-type $X^{s,b}$ structure. The other one shows a logarithmic divergence if we only use X_k which is the reason for us applying Y_k structure. See also the similar phenomenon in [15] for the complex-valued Benjamin-Ono equation.

Proposition 5.7. *Assume* $k\ge 200$. *Then there exist* $f_1\in X_1$, $f_k\in X_k$ *such that*

$$2^k\|\eta_k(\xi)(\tau-\omega(\xi)+i)^{-1}f_1*f_1*f_k\|_{X_k}\gtrsim k\|f_1\|_{X_1}\|f_1\|_{X_1}\|f_k\|_{X_k}. \quad (5.25)$$

Proof. From the proof of Proposition 5.1, we easily see that the worst interaction comes from the case that largest frequency component has a largest modulation. So we construct this case explicitly. Let $I=[1/2,1]$, and take

$$f_1(\xi,\tau)=\chi_I(\xi)\eta_1(\tau-\omega(\xi)),\ \ f_k(\xi,\tau)=\chi_{I_k}(\xi)\eta_k(\tau-\omega(\xi)).$$

From definition, we easily get $\|f_1\|_{X_1}\sim 1$ and $\|f_k\|_{X_k}\sim 2^{3k/2}$ and

$$2^k\|\eta_k(\xi)(\tau-\omega(\xi)+i)^{-1}f_1*f_1*f_k\|_{X_k}\gtrsim 2^k\sum_{j=0}^{k/2}2^{-j/2}\|1_{D_{k,j}}\cdot f_1*f_1*f_k\|_{L^2_{\xi,\tau}}.$$

On the other hand, we have for $j \le k/2$

$$\begin{aligned}
& 1_{D_{k,j}}(\xi,\tau)\cdot f_1 * f_1 * f_k \\
= & \int f_1(\xi_1,\tau_1)f_2(\xi_2,\tau_2)f_k(\xi-\xi_1-\xi_2,\tau-\tau_1-\tau_2)d\xi_1 d\xi_2 d\tau_1 d\tau_2 \\
= & \int \chi_I(\xi_1)\chi_I(\xi_2)\eta_1(\tau_1)\eta_1(\tau_2)\chi_{I_k}(\xi-\xi_1-\xi_2) \\
& \cdot\eta_k(\tau-\tau_1-\tau_2-\omega(\xi_1)-\omega(\xi_2)-\omega(\xi-\xi_1-\xi_2))d\xi_1 d\xi_2 d\tau_1 d\tau_2 \\
\gtrsim & \chi_{[\frac{2^{10}-1}{2^{10}}2^k,\frac{2^{10}+1}{2^{10}}2^k]}(\xi)\eta_j(\tau-\omega(\xi)).
\end{aligned}$$

Therefore, we get

$$2^k\sum_{j=0}^{k/2}2^{-j/2}\|1_{D_{k,j}}\cdot f_1 * f_1 * f_k\|_{L^2_{\xi,\tau}}\gtrsim k2^{3k/2}, \tag{5.26}$$

which completes the proof of the proposition. ■

Proposition 5.8. *For any $s\in\mathbb{R}$, there doesn't exists $b\in\mathbb{R}$ such that*

$$\|\partial_x(uvw)\|_{X^{s,b-1}}\lesssim \|u\|_{X^{s,b}}\|v\|_{X^{s,b}}\|w\|_{X^{s,b}}. \tag{5.27}$$

Proof. It is easy to see that the counterexample in the proof of Proposition 5.7 shows that (5.27) doesn't hold for $b=1/2$ with a $k^{1/2}$ divergence in (5.26). We assume now $b\neq 1/2$. By using Plancherel's equality, we get that (5.27) is equivalent to

$$\begin{aligned}
&\|\frac{\langle\xi\rangle^s\xi}{\langle\tau-\omega(\xi)\rangle^{1-b}}\int\frac{u(\xi_1,\tau_1)}{\langle\xi_1\rangle^s\langle\tau_1-\omega(\xi_1)\rangle^b}\frac{v(\xi_2,\tau_2)}{\langle\xi_2\rangle^s\langle\tau_2-\omega(\xi_2)\rangle^b} \\
&\quad\cdot\frac{w(\xi-\xi_1-\xi_2,\tau-\tau_1-\tau_2)}{\langle\xi-\xi_1-\xi_2\rangle^s\langle\tau-\tau_1-\tau_2-\omega(\xi-\xi_1-\xi_2)\rangle^b}d\tau_1 d\tau_2 d\xi_1 d\xi_2\|_{L^2_{\xi,\tau}} \\
&\lesssim \|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}.
\end{aligned}\tag{5.28}$$

Fix any dyadic number $N\gg 1$. Let

$$A=\{1/2\le\xi\le 10,\ |\tau|\le 1\}\text{ and }B=\{N/2\le\xi\le 2N,\ |\tau|\le 2^{10}\}.$$

Take

$$u(\xi,\tau)=v(\xi,\tau)=\chi_A(\xi,\tau-\omega(\xi)),\ w(\xi,\tau)=\chi_B(\xi,\tau-\omega(\xi)).$$

We easily see that $\|u\|_{L_2}=\|v\|_{L_2}\sim 1$ and $\|w\|_{L^2}\sim N^{1/2}$. Denote $f(\xi,\tau)=u*v*w(\xi,\tau+\omega(\xi))$. Then we have

$$\begin{aligned}
f(\xi,\tau) &= \int u(\xi_1,\tau_1)v(\xi_2,\tau_2)w(\xi-\xi_1-\xi_2,\tau+\omega(\xi)-\tau_1-\tau_2)d\xi_1 d\xi_2 d\tau_1 d\tau_2 \\
&= \int\chi_{\le 2^{10}}(\tau-\tau_1-\tau_2+\omega(\xi)-\omega(\xi-\xi_1-\xi_2)-\omega(\xi_1)-\omega(\xi_2)) \\
&\quad \chi_A(\xi_1,\tau_1)\chi_A(\xi_2,\tau_2)\chi_{[N/2,2N]}(\xi-\xi_1-\xi_2)d\xi_1 d\xi_2 d\tau_1 d\tau_2 \\
&= \int\chi_{\le 2^{10}}(\tau-\tau_1-\tau_2+2(\xi_1+\xi_2)\xi+(\xi_1-\xi_2)^2-\omega(\xi_1)-\omega(\xi_2)) \\
&\quad \chi_A(\xi_1,\tau_1)\chi_A(\xi_2,\tau_2)\chi_{[N/2,2N]}(\xi-\xi_1-\xi_2)d\xi_1 d\xi_2 d\tau_1 d\tau_2.
\end{aligned}$$

Therefore, fixing $M \gg 1$, we get for any $(\xi,\tau) \in [(M-1)N/M, (M+1)N/M] \times [-8N, -4N]$, then $\tau = -C_0\xi$ for some $2 \le C_0 \le 9$ and

$$f(\xi,\tau) \gtrsim \int \chi_A(\xi_1,\tau_1)\chi_A(\xi_2,\tau_2)\chi_{|\xi_1+\xi_2-C_0| \lesssim N^{-1}} d\xi_1 d\xi_2 d\tau_1 d\tau_2 \gtrsim N^{-1}.$$

Thus we see that the left-hand side of (5.28) is larger than N^b, while the right-hand side is $N^{1/2}$, which implies $b < 1/2$.

Similarly, by taking $B' = \{N/2 \le \xi \le 2N,\ N \le |\tau| \le N\}$ as before, we obtain that $b > 1/2$. Therefore we complete the proof of the proposition. ∎

6. Proof of Theorem 1.1

In this section we devote to prove Theorem 1.1 by using the standard fixed-point machinery. From Duhamel's principle, we get that the equation (1.1) is equivalent to the following integral equation:

$$u = W(t)\phi + \int_0^t W(t-t')(\partial_x(u^3)(t'))dt'. \tag{6.1}$$

We will mainly work on the following truncated version

$$u = \psi(t)W(t)\phi + \psi(t)\int_0^t W(t-t')(\partial_x[(\psi(t')u)^3](t'))dt', \tag{6.2}$$

where $\psi(t) = \eta_0(t)$ is a smooth cut-off function. Then we easily see that if u is a solution to (6.2) on $\mathbb{R}$, then u solves (6.1) on $t \in [-1,1]$. Our first lemma is on the estimate for the linear solution.

Lemma 6.1. *If $s \ge 0$ and $\phi \in H^s$ then*

$$\|\psi(t)\cdot(W(t)\phi)\|_{F^s} \le C\|\phi\|_{H^s}. \tag{6.3}$$

Proof. A direct computation shows that

$$\mathcal{F}[\psi(t)\cdot(W(t)\phi)](\xi,\tau) = \widehat{\phi}(\xi)\widehat{\psi}(\tau-\omega(\xi)).$$

In view of definition, it suffices to prove that if $k \in \mathbb{Z}_+$ then

$$\|\eta_k(\xi)\widehat{\phi}(\xi)\widehat{\psi}(\tau-\omega(\xi))\|_{Z_k} \le C\|\eta_k(\xi)\widehat{\phi}(\xi)\|_{L^2}. \tag{6.4}$$

Indeed, from definition we have

$$\begin{aligned}
\|\eta_k(\xi)\widehat{\phi}(\xi)\widehat{\psi}(\tau-\omega(\xi))\|_{Z_k} &\le \|\eta_k(\xi)\widehat{\phi}(\xi)\widehat{\psi}(\tau-\omega(\xi))\|_{X_k}\\
&\le C\sum_{j=0}^{\infty} 2^j \|\eta_k(\xi)\widehat{\phi}(\xi)\|_{L^2}\|\eta_j(\tau)\widehat{\psi}(\tau)\|_{L^2}\\
&\le C\|\eta_k(\xi)\widehat{\phi}(\xi)\|_{L^2},
\end{aligned}$$

which is (6.4) as desired. ∎

Next lemma is on the estimate for the retarded linear term. These estimates were also used in [14]. The only difference is that here we don't have special structure for the low frequency.

Lemma 6.2. *If $l, s \ge 0$ and $u \in \mathcal{S}(\mathbb{R}\times\mathbb{R})$ then*

$$\left\|\psi(t)\cdot\int_0^t W(t-s)(u(s))ds\right\|_{F^s} \le C\|u\|_{N^s}. \tag{6.5}$$

Proof. A straightforward computation shows that

$$\begin{aligned}&\mathcal{F}\left[\psi(t)\cdot\int_0^t W(t-s)(u(s))ds\right](\xi,\tau)\\&=c\int_{\mathbb{R}}\mathcal{F}(u)(\xi,\tau')\frac{\widehat{\psi}(\tau-\tau')-\widehat{\psi}(\tau-\omega(\xi))}{\tau'-\omega(\xi)}d\tau'.\end{aligned}$$

For $k\in\mathbb{Z}_+$ let $f_k(\xi,\tau')=\mathcal{F}(u)(\xi,\tau')\chi_k(\xi)(\tau'-\omega(\xi)+i)^{-1}$. For $f_k\in Z_k$ let

$$T(f_k)(\xi,\tau)=\int_{\mathbb{R}}f_k(\xi,\tau')\frac{\widehat{\psi}(\tau-\tau')-\widehat{\psi}(\tau-\omega(\xi))}{\tau'-\omega(\xi)}(\tau'-\omega(\xi)+i)d\tau'.$$

In view of the definitions, it suffices to prove that

$$\|T\|_{Z_k\to Z_k}\le C \text{ uniformly in } k\in Z_+, \tag{6.6}$$

which follows from the proof of Lemma 5.2 in [14]. ■

We prove a trilinear estimate in the following proposition which is an important component for using fixed-point argument.

Proposition 6.3. *Let $s\ge 1/2$. Then*

$$\begin{aligned}\|\partial_x(\psi(t)^3uvw)\|_{N^s}\quad\lesssim\quad&\|u\|_{F^s}\|v\|_{F^{\frac12}}\|w\|_{F^{\frac12}}\\&+\|u\|_{F^{\frac12}}\|v\|_{F^s}\|w\|_{F^{\frac12}}+\|u\|_{F^{\frac12}}\|v\|_{F^{\frac12}}\|w\|_{F^s}.\end{aligned}\tag{6.7}$$

Proof. For the simplicity of notation, we write $u=\psi(t)u$, $v=\psi(t)v$ and $w=\psi(t)w$. In view of definition, we get

$$\|\partial_x(uvw)\|_{N^s}^2=\sum_{k_4=0}^{\infty}2^{2sk_4}\|\eta_{k_4}(\xi)(\tau-\omega(\xi)+i)^{-1}\mathcal{F}(\partial_x(uvw))\|_{Z_k}^2.$$

Setting $f_{k_1}=\eta_{k_1}(\xi)\mathcal{F}(u)(\xi,\tau)$, $f_{k_2}=\eta_{k_2}(\xi)\mathcal{F}(v)(\xi,\tau)$, and $f_{k_3}=\eta_{k_3}(\xi)\mathcal{F}(w)(\xi,\tau)$, for $k_1,k_2,k_3\in\mathbb{Z}_+$, then we get

$$\begin{aligned}&2^{k_4}\|\eta_{k_4}(\xi)(\tau-\omega(\xi)+i)^{-1}\mathcal{F}(uvw)\|_{Z_{k_4}}\\&\lesssim\sum_{k_1,k_2,k_3\in\mathbb{Z}_+}2^{k_4}\|\eta_{k_4}(\xi)(\tau-\omega(\xi)+i)^{-1}f_{k_1}*f_{k_2}*f_{k_3}\|_{Z_{k_4}}.\end{aligned}$$

From symmetry it suffices to bound

$$\sum_{0\le k_1\le k_2\le k_3}2^{k_4}\|\eta_{k_4}(\xi)(\tau-\omega(\xi)+i)^{-1}f_{k_1}*f_{k_2}*f_{k_3}\|_{Z_{k_4}}.$$

Dividing the summation into the several parts, we get

$$\begin{aligned}&\sum_{k_1\le k_2\le k_3}2^{k_4}\|\eta_{k_4}(\xi)(\tau-\omega(\xi)+i)^{-1}f_{k_1}*f_{k_2}*f_{k_3}\|_{Z_{k_4}}\\\le\ &\sum_{j=1}^{6}\sum_{(k_1,k_2,k_3,k_4)\in A_j}2^{k_4}\|\eta_{k_4}(\xi)(\tau-\omega(\xi)+i)^{-1}f_{k_1}*f_{k_2}*f_{k_3}\|_{Z_{k_4}},\end{aligned}\tag{6.8}$$

where we denote

$$
\begin{aligned}
&A_1=\{0\leq k_1\leq k_2\leq k_3-10, k_3\geq 110, |k_4-k_3|\leq 5, |k_1-k_2|\leq 10\};\\
&A_2=\{0\leq k_1\leq k_2\leq k_3-10, k_3\geq 110, |k_4-k_3|\leq 5, k_1\leq k_2-5\};\\
&A_3=\{0\leq k_1\leq k_2\leq k_3, k_2\geq k_3-10, k_3\geq 110, |k_4-k_3|\leq 5, k_1\leq k_2-10\};\\
&A_4=\{0\leq k_1\leq k_2\leq k_3, k_1\geq k_3-30, k_3\geq 110, |k_4-k_3|\leq 5\};\\
&A_5=\{0\leq k_1\leq k_2\leq k_3, k_4\leq k_3-10, k_3\geq 110, |k_2-k_3|\leq 5\};\\
&A_6=\{0\leq k_1\leq k_2\leq k_3, \max(k_3,k_4)\leq 120\}.
\end{aligned}
$$

Noting that $\mathcal{F}^{-1}(f_{k_i})(x,t)$ is supported in $\mathbb{R}\times I$ with $|I|\lesssim 1$, we will apply Proposition 5.1-5.6 obtained in the last section to bound the six terms in (6.8). For example, for the first term, from Proposition 5.1, we have

$$
\begin{aligned}
&\Big\|2^{sk_4}\sum_{k_i\in A_1}2^{k_4}\|\eta_{k_4}(\xi)(\tau-\omega(\xi)+i)^{-1}f_{k_1}*f_{k_2}*f_{k_3}\|_{Z_{k_4}}\Big\|_{l^2_{k_4}}\\
\leq\ & C\Big\|2^{sk_4}\sum_{k_i\in A_1}2^{(k_1+k_2)/2}\|f_{k_1}\|_{Z_{k_1}}\|f_{k_2}\|_{Z_{k_2}}\|f_{k_3}\|_{Z_{k_3}}\Big\|_{l^2_{k_4}}\\
\leq\ & \|u\|_{F^{1/2}}\|v\|_{F^{1/2}}\|w\|_{F^s}.
\end{aligned}
$$

For the other terms we can handle them in the similar ways. Therefore we complete the proof of the proposition. ■

Now we prove Theorem 1.1. To begin with, we renormalize the data a bit via scaling. By the scaling (1.4), we see that if $s\geq 1/2$

$$
\begin{aligned}
&\|\phi_\lambda\|_{L^2}=\|\phi\|_{L^2},\\
&\|\phi_\lambda\|_{\dot H^s}=\lambda^{-s}\|\phi\|_{\dot H^s}.
\end{aligned}
$$

From the assumption $\|\phi\|_{L^2}\ll 1$, thus we can first restrict ourselves to considering (1.1) with data ϕ satisfying

$$\|\phi\|_{H^s}=r\ll 1. \tag{6.9}$$

This indicates the reason why we assume that $\|\phi\|_{L^2}\ll 1$.

Define the operator

$$\Phi_\phi(u)=\psi(t)W(t)\phi+\psi(t)\int_0^t W(t-t')(\partial_x((\psi(t')u)^3)(t'))dt',$$

and we will prove that $\Phi_\phi(\cdot)$ is a contraction mapping from

$$\mathcal{B}=\{w\in F^s:\ \|w\|_{F^s}\leq 2cr\} \tag{6.10}$$

into itself. From Lemma 6.1, 6.2 and Proposition 6.3 we get if $w\in\mathcal{B}$, then

$$
\begin{aligned}
\|\Phi_\phi(w)\|_{F^s}\ &\leq\ c\|\phi\|_{H^s}+\|\partial_x(\psi(t)^3w^3(\cdot,t))\|_{N^s}\\
&\leq\ cr+c\|w\|_{F^s}^3\leq cr+c(2cr)^3\leq 2cr,
\end{aligned}\tag{6.11}
$$

provided that r satisfies $8c^3r^2\leq 1/2$. Similarly, for $w,h\in\mathcal{B}$

$$
\begin{aligned}
\|\Phi_\phi(w)-\Phi_\phi(h)\|_{F^s}\ &\leq\ c\left\|L\partial_x(\psi^3(\tau)(w^3(\tau)-h^3(\tau)))\right\|_{F^s}\\
&\leq\ c(\|w\|_{F^s}^2+\|h\|_{F^s}^2)\|w-h\|_{F^s}\\
&\leq\ 8c^3r^2\|w-h\|_{F^s}\leq\frac{1}{2}\|w-h\|_{F^s}.
\end{aligned}\tag{6.12}
$$

Thus $\Phi_\phi(\cdot)$ is a contraction. Therefore, there exists a unique $u \in \mathcal{B}$ such that

$$u = \psi(t)W(t)\phi + \psi(t)\int_0^t W(t-t')(\partial_x[(\psi(t')u)^3](t'))dt'.$$

Hence u solves the integral equation (6.1) in the time interval $[-1,1]$.

Part (c) of Theorem 1.1 follows from the scaling (1.4), Lemma 3.4 and Proposition 6.3. We prove now part (b). For the real-valued case, according to Theorem 1.2 in [18], it suffices to prove that if $s>1$ then

$$\partial_x u \in L^4_{t\in[0,T]}L^\infty_x.$$

Indeed, this follows from the fact that $u \in F^s(T)$ and $(4,\infty)$ is an admissible pair and Lemma 3.4. For the complex-valued case, we have some weak uniqueness. From the proof we see u is unique in the following set

$$B(u_0) = \left\{ u : \begin{array}{l} u = \lambda^{1/2}\widetilde{u}(\lambda^2 t, \lambda x) \text{ in } [-T,T] \text{ for some } \lambda = \lambda(\|\phi\|_{H^s}) \gg 1 \\ \text{and a solution } \widetilde{u} \text{ to (6.2) satisfying } \|\widetilde{u}\|_{F^s} \lesssim \lambda^{-s} \end{array} \right\} \tag{6.13}$$

Therefore, we complete the proof of Theorem 1.1.

Remark 6.4. For the real-valued case, the uniqueness actually holds in $F^s(T)$ by the uniqueness in [25] and Lemma 3.4. From the proof we see the L^2 norm smallness condition is due to the $high \times low \to high$ interaction where both low frequency are around 0. It is also due to this interaction that one can not apply the methods as in the second part. This bad interaction is removed via gauge transformation in the previous results. On the other hand, one may also remove the smallness condition by performing a gauge transformation as following

$$v(x,t) = e^{-(i/2)\int_{-\infty}^x (P_{\lesssim 1}u(y,t))^2 dy} P_+ P_{\gg 1} u(x,t),$$

and using the similar methods in [14].

7. Short-time Trilinear Estimates

In this section we devote to prove some dyadic trilinear estimates in the spaces $F_k,\ N_k$. For $k \in \mathbb{Z}$ and $j \in \mathbb{Z}$ let

$$\widetilde{D}_{k,j} = \{(\xi,\tau) : \xi \in I_k, |\tau - \omega(\xi)| \le 2^j\}. \tag{7.1}$$

Note that $\dot{D}_{k,j} \subset \widetilde{D}_{k,j}$ for any $k,j \in \mathbb{Z}$.

Proposition 7.1. *If $k_4 \ge 20$, $|k_3 - k_4| \le 5$, $k_1 \le k_2 \le k_3 - 10$, then*

$$\|R_{k_4}\partial_x(u_{k_1}v_{k_2}w_{k_3})\|_{N_{k_4}} \le C\min(2^{k_1/2}, \langle k_2\rangle)\|u_{k_1}\|_{F_{k_1}}\|v_{k_2}\|_{F_{k_2}}\|w_{k_3}\|_{F_{k_3}}. \tag{7.2}$$

Proof. Using the definitions and (3.11), we get that the left-hand side of (7.2) is dominated by

$$\begin{aligned} C\sup_{t_k\in\mathbb{R}} &\|(\tau - \omega(\xi) + i2^{k_4})^{-1} \cdot 2^{k_4}1_{I_k}(\xi) \cdot \mathcal{F}[u_{k_1}\eta_0(2^{k_4-2}(t-t_k))] \\ &* \mathcal{F}[v_{k_2}\eta_0(2^{k_4-2}(t-t_k))] * \mathcal{F}[w_{k_3}\eta_0(2^{k_4-2}(t-t_k))]\|_{B_k}. \end{aligned} \tag{7.3}$$

It suffices to prove that if $j_i \ge k_4$ and $f_{k_i,j_i} : \mathbb{R}^2 \to \mathbb{R}_+$ are supported in $\widetilde{D}_{k_i,j_i}$ for $i = 1,2,3$, then

$$\begin{aligned} &2^{k_4}\sum_{j_4\ge k_4} 2^{-j_4/2}\|1_{\widetilde{D}_{k_4,j_4}} \cdot (f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3})\|_{L^2} \\ \le\ & C\min(2^{k_1/2}, \langle k_2\rangle)2^{j_1/2}\|f_{k_1,j_1}\|_{L^2}2^{j_2/2}\|f_{k_2,j_2}\|_{L^2}2^{j_3/2}\|f_{k_3,j_3}\|_{L^2}. \end{aligned} \tag{7.4}$$

We assume first (7.4). Let $f_{k_1}=\mathcal{F}[u_{k_1}\cdot\eta_0(2^{k_4-2}(t-t_k))]$, $f_{k_2}=\mathcal{F}[v_{k_2}\cdot\eta_0(2^{k_4-2}(t-t_k))]$ and $f_{k_3}=\mathcal{F}[w_{k_3}\cdot\eta_0(2^{k_4-2}(t-t_k))]$. Then from the definition of B_k we get that (7.3) is dominated by

$$\sup_{t_k\in\mathbb{R}} 2^{k_4}\sum_{j_4=0}^{\infty}2^{j_4/2}\sum_{j_1,j_2,j_3\geq k_4}\|(2^{j_4}+i2^{k_4})^{-1}1_{\dot{D}_{k_4,j_4}}\cdot f_{k_1,j_1}*f_{k_2,j_2}*f_{k_3,j_3}\|_{L^2}, \quad (7.5)$$

where $f_{k_i,j_i}=f_{k_i}(\xi,\tau)\eta_{j_i}(\tau-\omega(\xi))$ for $j_i>k_4$ and $f_{k_i,k_4}=f_{k_i}(\xi,\tau)\eta_{\leq k_4}(\tau-\omega(\xi))$, $i=1,2,3$. For the summation on the terms $j_4<k_4$ in (7.5), we get from the fact $1_{D_{k_4,j_4}}\leq 1_{\widetilde{D}_{k_4,j_4}}$ that

$$\begin{aligned}&\sup_{t_k\in\mathbb{R}} 2^{k_4}\sum_{j_4<k_4}2^{j_4/2}\sum_{j_1,j_2,j_3\geq k_4}\|(2^{j_4}+i2^{k_4})^{-1}1_{\dot{D}_{k_4,j_4}}\cdot f_{k_1,j_1}*f_{k_2,j_2}*f_{k_3,j_3}\|_{L^2}\\ &\lesssim \sup_{t_k\in\mathbb{R}}\sum_{j_1,j_2,j_3\geq k_4}2^{k_4/2}\|1_{\widetilde{D}_{k_4,k_4}}\cdot f_{k_1,j_1}*f_{k_2,j_2}*f_{k_3,j_3}\|_{L^2}. \qquad (7.6)\end{aligned}$$

From the fact that f_{k_i,j_i} is supported in $\widetilde{D}_{k_i,j_i}$ for $i=1,2,3$ and using (7.4), we get that

$$\begin{aligned}&\sup_{t_k\in\mathbb{R}}\sum_{j_1,j_2,j_3\geq k_4}2^{k_4}\sum_{j_4\geq k_4}2^{-j_4/2}\|1_{\widetilde{D}_{k_4,j_4}}\cdot f_{k_1,j_1}*f_{k_2,j_2}*f_{k_3,j_3}\|_{L^2}\\ &\lesssim \sup_{t_k\in\mathbb{R}}\min(2^{k_1/2},\langle k_2\rangle)\sum_{j_1,j_2,j_3\geq k_4}2^{j_1/2}\|f_{k_1,j_1}\|_{L^2}2^{j_2/2}\|f_{k_2,j_2}\|_{L^2}2^{j_3/2}\|f_{k_3,j_3}\|_{L^2}.\end{aligned}$$

Thus using (3.10) and (3.11) we obtain (7.2), as desired.

To prove (7.4), we consider first the case $|k_1-k_2|\leq 5$. If $k_2\geq 0$, it follows from Corollary 4.2 (b) and Remark 4.3 that

$$\begin{aligned}&2^{k_4}\sum_{j_4\geq k_4}2^{-j_4/2}\|1_{\widetilde{D}_{k_4,j_4}}\cdot(f_{k_1,j_1}*f_{k_2,j_2}*f_{k_3,j_3})\|_{L^2}\\ \lesssim\;& 2^{k_4}\sum_{j_4\geq k_4+k_2}2^{-j_4/2}2^{(j_1+j_2+j_3)/2}2^{-k_4/2}2^{k_1/2}\prod_{i=1}^{3}\|f_{k_i,j_i}\|_{L^2}\\ &+2^{k_4}\sum_{k_4\leq j_4\leq k_4+k_2}2^{-j_4/2}2^{(j_1+j_2+j_3+j_4)/2}2^{-k_4}\prod_{i=1}^{3}\|f_{k_i,j_i}\|_{L^2}\\ \lesssim\;&(1+k_2)2^{(j_1+j_2+j_3)/2}\prod_{i=1}^{3}\|f_{k_i,j_i}\|_{L^2}, \qquad (7.7)\end{aligned}$$

which is (7.4) as desired. If $k_2<0$, then from Corollary 4.2 (a) and Remark 4.3 we get that

$$\begin{aligned}&2^{k_4}\sum_{j_4\geq k_4}2^{-j_4/2}\|1_{\widetilde{D}_{k_4,j_4}}\cdot(f_{k_1,j_1}*f_{k_2,j_2}*f_{k_3,j_3})\|_{L^2}\\ \lesssim\;& 2^{k_4}\sum_{j_4\geq k_4}2^{-j_4/2}2^{(j_1+j_2+j_3)/2}2^{-j_3/2}2^{k_1/2}2^{k_2/2}\prod_{i=1}^{3}\|f_{k_i,j_i}\|_{L^2}\\ \lesssim\;&2^{k_1/2}2^{(j_1+j_2+j_3)/2}\prod_{i=1}^{3}\|f_{k_i,j_i}\|_{L^2}, \qquad (7.8)\end{aligned}$$

which is (7.4) as desired. We assume now $k_1 < k_2 - 5$. If $k_2 < 0$, then arguing as in (7.8) we get (7.4) as desired. If $k_2 > 0$, then (7.7) also holds in this case. On the other hand, by checking the support properties of the function f_{k_i,j_i}, $i = 1, 2, 3$, we get that $1_{\widetilde{D}_{k_4,j_4}} \cdot (f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3}) \equiv 0$ unless $j_{max} \geq k_4 + k_2 - 20$. For the summation on the terms $j_4 > k_4 + k_2 - 30$ in (7.4), we have

$$\begin{aligned}
& 2^{k_4} \sum_{j_4 \geq k_4+k_2-30} 2^{-j_4/2} \|1_{\widetilde{D}_{k_4,j_4}} \cdot (f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3})\|_{L^2} \\
\lesssim\ & 2^{k_4} \sum_{j_4 \geq k_4+k_2-30} 2^{-j_4/2} 2^{(j_1+j_2+j_3)/2} 2^{-j_3/2} 2^{k_1/2} 2^{k_2/2} \prod_{i=1}^{3} \|f_{k_i,j_i}\|_{L^2} \\
\lesssim\ & 2^{k_1/2} 2^{(j_1+j_2+j_3)/2} \prod_{i=1}^{3} \|f_{k_i,j_i}\|_{L^2}. \qquad (7.9)
\end{aligned}$$

For the summation on the terms $j_4 < k_4 + k_2 - 30$, we have $j_4 \leq j_{med}$. Thus using Corollary 4.2 (a), then we get

$$\begin{aligned}
& 2^{k_4} \sum_{j_4 < k_4+k_2-30} 2^{-j_4/2} \|1_{\widetilde{D}_{k_4,j_4}} \cdot (f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3})\|_{L^2} \\
\lesssim\ & 2^{k_4} \sum_{k_4 \leq j_4 < k_4+k_2-30} 2^{-j_4/2} 2^{(j_1+j_2+j_3)/2} 2^{-j_{max}/2} 2^{k_1/2} 2^{k_2/2} \prod_{i=1}^{3} \|f_{k_i,j_i}\|_{L^2} \\
\lesssim\ & 2^{k_1/2} 2^{(j_1+j_2+j_3)/2} \prod_{i=1}^{3} \|f_{k_i,j_i}\|_{L^2}. \qquad (7.10)
\end{aligned}$$

Therefore, we complete the proof of the proposition. ■

Proposition 7.2. *If* $k_4 \geq 20$, $|k_3 - k_4| \leq 5$, $k_3 - 10 \leq k_2 \leq k_3$ *and* $k_1 \leq k_2 - 20$, *then we have*

$$\|R_{k_4} \partial_x (u_{k_1} v_{k_2} w_{k_3})\|_{N_{k_4}} \leq C \min(2^{k_1/2}, 1) \|u_{k_1}\|_{F_{k_1}} \|v_{k_2}\|_{F_{k_2}} \|w_{k_3}\|_{F_{k_3}}. \qquad (7.11)$$

Proof. As in the proof of Proposition 7.1, using (3.10) and (3.11), we see that it suffices to prove that if $j_1, j_2, j_3 \geq k_4$, and $f_{k_i,j_i} : \mathbb{R}^2 \to \mathbb{R}_+$ are supported in $\widetilde{D}_{k_i,j_i}$, $i = 1, 2, 3$, then

$$\begin{aligned}
& 2^{k_4} \sum_{j_4 \geq k_4} 2^{-j_4/2} \|1_{\widetilde{D}_{k_4,j_4}} \cdot (f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3})\|_{L^2} \\
& \leq C \min(2^{k_1/2}, 1) \cdot 2^{j_1/2} \|f_{k_1,j_1}\|_{L^2} \cdot 2^{j_2/2} \|f_{k_2,j_2}\|_{L^2} \cdot 2^{j_3/2} \|f_{k_3,j_3}\|_{L^2}. \qquad (7.12)
\end{aligned}$$

Since in the area $\{|\xi_i| \in \widetilde{I}_{k_i}, i = 1, 2, 3\} \cap \{|\xi_1 + \xi_2 + \xi_3| \in I_{k_4}\}$

$$|\Omega(\xi_1, \xi_2, \xi_3)| \sim 2^{2k_4},$$

then by checking the support properties, we get $1_{\widetilde{D}_{k_4,j_4}} \cdot (f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3}) \equiv 0$ unless $j_{max} \geq 2k_4 - 30$. It follows from Corollary 4.2 (a) that the left-hand side of (7.12) is bounded by

$$2^{k_4} \sum_{j_4 \geq k_4} 2^{-j_4/2} 2^{(j_1+j_2+j_3+j_4)/2} 2^{-j_{max}/2} 2^{-j_{sub}/2} 2^{k_1/2} 2^{k_2/2} \prod_{i=1}^{3} \|f_{k_i,j_i}\|_{L^2}. \qquad (7.13)$$

Then we get the bound (7.12) by considering either $j_4 = j_{max}$ or $j_4 \neq j_{max}$. ■

Proposition 7.3. *If $k_4 \geq 20$, $|k_3-k_4| \leq 5$, $k_3-10 \leq k_2 \leq k_3$ and $k_2-30 \leq k_1 \leq k_2$, then we have*

$$\|R_{k_4}\partial_x(u_{k_1}v_{k_2}w_{k_3})\|_{N_{k_4}} \leq C2^{k_3/2}\|u_{k_1}\|_{F_{k_1}}\|v_{k_2}\|_{F_{k_2}}\|w_{k_3}\|_{F_{k_3}}. \tag{7.14}$$

Proof. As in the proof of Proposition 7.1, using (3.10) and (3.11), it suffices to prove that if $j_1, j_2, j_3 \geq k_4$, and $f_{k_i,j_i} : \mathbb{R}^2 \to \mathbb{R}_+$ are supported in $\widetilde{D}_{k_i,j_i}$, $i = 1, 2, 3$, then

$$\begin{aligned}&2^{k_4}\sum_{j_4\geq k_4} 2^{-j_4/2}\|1_{\widetilde{D}_{k_4,j_4}} \cdot (f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3})\|_{L^2}\\ &\leq C2^{k_4/2} \cdot 2^{j_1/2}\|f_{k_1,j_1}\|_{L^2} \cdot 2^{j_2/2}\|f_{k_2,j_2}\|_{L^2} \cdot 2^{j_3/2}\|f_{k_3,j_3}\|_{L^2},\end{aligned}$$

which follows immediately from Corollary 4.2 (c). ■

Proposition 7.4. *If $k_3 \geq 20$, $|k_3 - k_2| \leq 5$, $k_2 - 10 \leq k_1 \leq k_2$ and $k_4 \leq k_1 - 30$, then we have*

$$\|R_{k_4}\partial_x(u_{k_1}v_{k_2}w_{k_3})\|_{N_{k_4}} \leq C\min(2^{k_4}, 1)|k_2|\|u_{k_1}\|_{F_{k_1}}\|v_{k_2}\|_{F_{k_2}}\|w_{k_3}\|_{F_{k_3}}. \tag{7.15}$$

Proof. Let $\gamma : \mathbb{R} \to [0, 1]$ denote a smooth function supported in $[-1, 1]$ with the property that

$$\sum_{n\in\mathbb{Z}} \gamma^3(x - n) \equiv 1, \quad x \in \mathbb{R}.$$

Using the definitions, the left-hand side of (7.15) is dominated by

$$\begin{aligned}C\sup_{t_k\in\mathbb{R}}\Big\|&(\tau - \omega(\xi) + i2^{k_{4+}})^{-1} \cdot 2^{k_4}1_{I_{k_4}}(\xi) \cdot \sum_{|m|\leq C2^{k_2-k_{4+}}}\\ &\mathcal{F}[u_{k_1}\eta_0(2^{k_{4+}}(t - t_k))\gamma(2^{k_2}(t - t_k) - m)] *\\ &\mathcal{F}[v_{k_2}\eta_0(2^{k_{4+}}(t - t_k))\gamma(2^{k_2}(t - t_k) - m)] *\\ &\mathcal{F}[w_{k_3}\eta_0(2^{k_{4+}}(t - t_k))\gamma(2^{k_2}(t - t_k) - m)]\Big\|_{B_k}.\end{aligned}$$

In view of the definitions, (3.10) and (3.11), it suffices to prove that if $j_1, j_2, j_3 \geq k_2$, and $f_{k_i,j_i} : \mathbb{R}^3 \to \mathbb{R}_+$ are supported in $\widetilde{D}_{k_i,j_i}$, $i = 1, 2, 3$, then

$$\begin{aligned}&2^{k_4}2^{k_2-k_{4+}}\sum_{j_4\geq k_{4+}} 2^{-j_4/2}\|1_{\widetilde{D}_{k_4,j_4}} \cdot (f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3})\|_{L^2}\\ &\leq C\min(2^{k_4}, 1)|k_2| \cdot 2^{j_1/2}\|f_{k_1,j_1}\|_{L^2} \cdot 2^{j_2/2}\|f_{k_2,j_2}\|_{L^2} \cdot 2^{j_3/2}\|f_{k_3,j_3}\|_{L^2}. \end{aligned}\tag{7.16}$$

From the same argument as in Proposition 7.2, we get $j_{max} \geq 2k_2 - 30$. Then (7.16) follows from Corollary 4.2. ■

Proposition 7.5. *If $k_3 \geq 20$, $|k_3 - k_2| \leq 5$, and $k_1, k_4 \leq k_2 - 10$, then*

$$\|R_{k_4}\partial_x(u_{k_1}v_{k_2}w_{k_3})\|_{N_{k_4}} \leq C\min(2^{k_1/2}, 1)|k_2|\|u_{k_1}\|_{F_{k_1}}\|v_{k_2}\|_{F_{k_2}}\|w_{k_3}\|_{F_{k_3}}. \tag{7.17}$$

Proof. As in the proof of Proposition 7.4, it suffices to prove that if $j_1, j_2, j_3 \geq k_2$, and $f_{k_i,j_i} : \mathbb{R}^3 \to \mathbb{R}_+$ are supported in $\widetilde{D}_{k_i,j_i}$, $i = 1, 2, 3$, then

$$\begin{aligned}&2^{k_4}2^{k_2-k_{4+}}\sum_{j_4\geq k_{4+}} 2^{-j_4/2}\|1_{\widetilde{D}_{k_4,j_4}} \cdot (f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3})\|_{L^2}\\ &\leq C\min(2^{k_1/2}, 1)|k_2| \cdot 2^{j_1/2}\|f_{k_1,j_1}\|_{L^2} \cdot 2^{j_2/2}\|f_{k_2,j_2}\|_{L^2} \cdot 2^{j_3/2}\|f_{k_3,j_3}\|_{L^2}. \end{aligned}\tag{7.18}$$

In proving (7.18) we may assume $j_4 \leq 10k_2$ in the summation of (7.18), otherwise we use Corollary 4.2 (a). Using Corollary 4.2 (a) for $k_1 \leq 0$, else using Corollary 4.2 (b), then we get

$$\begin{aligned}
&2^{k_4}2^{k_2-k_{4+}}\sum_{j_4\geq k_{4+}} 2^{-j_4/2}\|1_{\widetilde{D}_{k_4,j_4}}\cdot(f_{k_1,j_1}*f_{k_2,j_2}*f_{k_3,j_3})\|_{L^2}\\
&\leq C\min(2^{k_1/2},1)|k_2|2^{j_1/2}\|f_{k_1,j_1}\|_{L^2}2^{j_2/2}\|f_{k_2,j_2}\|_{L^2}2^{j_3/2}\|f_{k_3,j_3}\|_{L^2}.
\end{aligned}$$

Therefore, we complete the proof of the proposition. ■

Proposition 7.6. *If $k_1, k_2, k_3, k_4 \leq 200$, then*

$$\|R_{k_4}\partial_x(u_{k_1}v_{k_2}w_{k_3})\|_{N_{k_4}} \leq C2^{k_{min}/2}2^{k_{thd}/2}\|u_{k_1}\|_{F_{k_1}}\|v_{k_2}\|_{F_{k_2}}\|w_{k_3}\|_{F_{k_3}}. \tag{7.19}$$

Proof. This follows immediately from the definitions, Corollary 4.2 (a), Remark 4.3 and (3.10) and (3.11). ■

8. Energy Estimates

In this section we prove an energy estimate by using I-method [8], following some ideas in [26]. For the difference equation of two modified Benjamin-Ono equations, we don't know how to prove a similar energy estimate due to the lack of symmetry. That's why we can only solve the half problem of Conjecture.

Proposition 8.1. *Assume that $T \in (0,1]$ and $u \in C([-T,T] : H^\infty)$ is a real-valued solution of the initial value problem*

$$\begin{cases} u_t + \mathcal{H}u_{xx} = u^2u_x, \ (x,t) \in \mathbb{R}\times(-T,T);\\ u(x,0) = \phi(x), \end{cases} \tag{8.1}$$

Then, for $0 \leq l < 1/4$ and $s > 1/4$, there exists $\delta_0 > 0$ such that if $\|u\|_{E^{l,s}(T)} \leq \delta_0$ then we have

$$\|u\|^2_{E^{l,s}(T)} \lesssim \|\phi\|^2_{\dot{H}^l\cap\dot{H}^s} + \|u\|^4_{F^{\frac14-,\frac14+}(T)}\cdot\|u\|^2_{F^{l,s}(T)}. \tag{8.2}$$

The following definition was first introduced in [26].

Definition 8.2. Let $s \in \mathbb{R}$ and $\epsilon > 0$. Then S^s_ϵ is the class of spherical symmetric symbols with the following properties:

(i) symbol regularity,

$$|\partial^\alpha a(\xi)| \lesssim a(\xi)(1+\xi^2)^{-\alpha/2}$$

(ii) decay at infinity, for $|\xi| \gg 1$,

$$s \leq \frac{\log a(\xi)}{\log(1+\xi^2)} \leq s+\epsilon, \quad s-\epsilon \leq \frac{d\log a(\xi)}{d\log(1+\xi^2)} \leq s+\epsilon.$$

Assume $u \in C([-T,T] : H^\infty)$ solves (8.1) and $a \in S^s_\epsilon$. Denote $A(D) = \mathcal{F}^{-1}a(\xi)\mathcal{F}$. We first set

$$E_0(u) = (A(D)u, u) = \int_{\xi_1+\xi_2=0} a(\xi_1)\widehat{u}(\xi_1)\widehat{u}(\xi_2).$$

Using the equation (8.1) and noting that $a(\xi)$ is even while $\omega(\xi)$ is odd, then we easily get that

$$\begin{aligned}&\frac{d}{dt}E_0(u)=R_4(u)\\&=-\frac{1}{6}\int_{\Gamma_4} i[\xi_1 a(\xi_1)+\xi_2 a(\xi_2)+\xi_3 a(\xi_3)+\xi_4 a(\xi_4)]\widehat{u}(\xi_1)\widehat{u}(\xi_2)\widehat{u}(\xi_3)\widehat{u}(\xi_4),\end{aligned}$$

where for $k\in\mathbb{N}$, we denote

$$\Gamma_k=\{\xi_1+\xi_2+\ldots+\xi_k=0\}.$$

Following the idea of I-method, we define a multi-linear correction term to achieve a cancelation

$$E_1(u)=\int_{\Gamma_4} b_4(\xi_1,\xi_2,\xi_3,\xi_4)\widehat{u}(\xi_1)\widehat{u}(\xi_2)\widehat{u}(\xi_3)\widehat{u}(\xi_4),$$

where b_4 will be determined soon. Again using the equation (1.1), we get

$$\begin{aligned}&\frac{d}{dt}E_1(u)=R_6(u)\\&+\int_{\Gamma_4} ib_4(\xi_1,\xi_2,\xi_3,\xi_4)[\omega(\xi_1)+\omega(\xi_2)+\omega(\xi_3)+\omega(\xi_4)]\widehat{u}(\xi_1)\widehat{u}(\xi_2)\widehat{u}(\xi_3)\widehat{u}(\xi_4),\end{aligned}$$

where

$$R_6(u)=C\int_{\Gamma_6} b_4(\xi_1,\xi_2,\xi_3,\xi_4+\xi_5+\xi_6)(\xi_4+\xi_5+\xi_6)\prod_{j=1}^{6}\widehat{u}(\xi_j).$$

To achieve the cancelation of the quadrilinear form we define b_4 on Γ_4 by

$$b_4(\xi_1,\xi_2,\xi_3,\xi_4)=C\frac{\xi_1 a(\xi_1)+\xi_2 a(\xi_2)+\xi_3 a(\xi_3)+\xi_4 a(\xi_4)}{\omega(\xi_1)+\omega(\xi_2)+\omega(\xi_3)+\omega(\xi_4)}.$$

Thus we get

$$\frac{d}{dt}(E_0(u)+E_1(u))=R_6(u).$$

Proposition 8.3. *Assume that $a\in S^s_\epsilon$. Then for each dyadic $\lambda\leq\alpha\leq\mu$ there is an extension of b_4 from the diagonal set*

$$\{(\xi_1,\xi_2,\xi_3,\xi_4)\in\Gamma_4,|\xi_1|\sim\lambda,|\xi_2|\sim\alpha,|\xi_3|,|\xi_4|\sim\mu\}$$

to the full dyadic set

$$\{(\xi_1,\xi_2,\xi_3,\xi_4)\in\mathbb{R}_4,|\xi_1|\sim\lambda,|\xi_2|\sim\alpha,|\xi_3|,|\xi_4|\sim\mu\}$$

which satisfies

$$|b_4(\xi_1,\xi_2,\xi_3,\xi_4)|\lesssim a(\mu)\mu^{-1} \tag{8.3}$$

and

$$\sum_{j=1}^{4}|\partial_j b_4(\xi_1,\xi_2,\xi_3,\xi_4)|\lesssim a(\alpha)\mu^{-1}+a(\mu)\mu^{-2}. \tag{8.4}$$

Proof. From symmetry we may assume $|\xi_1| \leq |\xi_2| \leq |\xi_3| \leq |\xi_4|$ and $\xi_3 > 0, \xi_4 < 0$. We first consider the case that $\xi_1\xi_2 > 0$, say $\xi_1, \xi_2 > 0$. Then $\omega(\xi_1)+\omega(\xi_2)+\omega(\xi_3)+\omega(\xi_4) = -2(\xi_1\xi_2 + \xi_2\xi_3 + \xi_1\xi_3)$. Thus in Γ_4 we have

$$Cb_4(\xi_1, \xi_2, \xi_3, \xi_4) = \frac{\xi_1 a(\xi_1) + \xi_2 a(\xi_2)}{\xi_1\xi_2 + (\xi_2 + \xi_1)\xi_3} + \frac{\xi_3 a(\xi_3) + \xi_4 a(\xi_4)}{\xi_1\xi_2 + \xi_2\xi_3 + \xi_1\xi_3}.$$

Using $\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0$ we get

$$\begin{aligned}&\frac{\xi_3 a(\xi_3) + \xi_4 a(\xi_4)}{\xi_1\xi_2 + \xi_2\xi_3 + \xi_1\xi_3}\\ &= \frac{\xi_1\xi_2}{\xi_1\xi_2 + \xi_2\xi_3 + \xi_1\xi_3}\frac{\xi_3 a(\xi_3) + \xi_4 a(\xi_4)}{\xi_3(\xi_3 + \xi_4)} - \frac{\xi_3 a(\xi_3) + \xi_4 a(\xi_4)}{\xi_3(\xi_3 + \xi_4)}.\end{aligned}$$

Therefore, we extend b_4 by setting

$$\begin{aligned}Cb_4(\xi_1, \xi_2, \xi_3, \xi_4) = {}& \frac{\xi_1 a(\xi_1) + \xi_2 a(\xi_2)}{\xi_1\xi_2 + \xi_2\xi_3 + \xi_1\xi_3}\\ &+\frac{\xi_1\xi_2}{\xi_1\xi_2 + \xi_2\xi_3 + \xi_1\xi_3}\frac{\xi_3 a(\xi_3) + \xi_4 a(\xi_4)}{\xi_3(\xi_3 + \xi_4)} - \frac{\xi_3 a(\xi_3) + \xi_4 a(\xi_4)}{\xi_3(\xi_3 + \xi_4)}.\end{aligned} \tag{8.5}$$

It is easy to see from the properties of $a(\xi)$ that

$$|b_4(\xi_1, \xi_2, \xi_3, \xi_4)| \lesssim a(\mu)\mu^{-1}.$$

It remains to check the derivatives. We only consider $|\partial_1 b_4|$, since the others can be handled in the similar ways. For $|\partial_1 b_4|$ it suffices to consider the first term on the right-hand side of (8.5). Direct computations show that

$$\partial_{\xi_1}\left(\frac{\xi_1 a(\xi_1) + \xi_2 a(\xi_2)}{\xi_1\xi_2 + (\xi_2 + \xi_1)\xi_3}\right) = \frac{[a(\xi_1) - a(\xi_2)]\xi_2\xi_3 - \xi_2^2 a(\xi_2)}{(\xi_1\xi_2 + (\xi_2 + \xi_1)\xi_3)^2} + \frac{a'(\xi_1)\xi_1}{\xi_1\xi_2 + (\xi_2 + \xi_1)\xi_3},$$

which satisfies (8.4) as desired.

We consider now $\xi_1\xi_2 < 0$, say $\xi_1 < 0, \xi_2 > 0$. Thus we get $\omega(\xi_1) + \omega(\xi_2) + \omega(\xi_3) + \omega(\xi_4) = \xi_1^2 - \xi_2^2 - \xi_3^2 + \xi_4^2 = (\xi_1 + \xi_2)(\xi_1 + \xi_3)$. We will extend b_4 in the following cases.

(a) $\lambda \ll \mu$, $\alpha \leq \mu$. Then the extension of b_4 is defined using the formula

$$b_4(\xi_1, \xi_2, \xi_3, \xi_4) = \frac{\xi_1 a(\xi_1) + \xi_2 a(\xi_2)}{(\xi_1 + \xi_2)(\xi_1 + \xi_3)} - \frac{\xi_3 a(\xi_3) + \xi_4 a(\xi_4)}{(\xi_3 + \xi_4)(\xi_1 + \xi_3)}.$$

Since $\lambda \ll \mu$, we see that $|\xi_1 + \xi_3| \sim \mu$. By using the properties of $a(\xi)$ we see (8.3) and (8.4) are satisfied as desired.

(b) $\lambda \sim \mu$. Then the extension of b_4 is defined using the formula

$$b_4(\xi_1, \xi_2, \xi_3, \xi_4) = \frac{\xi_1 a(\xi_1) + \xi_2 a(\xi_2) + \xi_3 a(\xi_3) - (\xi_1 + \xi_2 + \xi_3)a(\xi_1 + \xi_2 + \xi_3)}{(\xi_1 + \xi_2)(\xi_1 + \xi_3)}.$$

To check the properties, setting

$$q(\xi_1, \xi_2) = \frac{\xi_1 a(\xi_1) + \xi_2 a(\xi_2)}{\xi_1 + \xi_2},$$

then we get that

$$b_4(\xi_1, \xi_2, \xi_3, \xi_4) = \frac{q(\xi_1, \xi_2) - q(\xi_1 - (\xi_1 + \xi_3), \xi_2 + (\xi_1 + \xi_3))}{\xi_1 + \xi_3},$$

from which we easily verify (8.3) and (8.4). ∎

In view of the definition, for Proposition 8.1 we are mainly concerned with the control of the energy in high frequency. From the definition we see that if $a \in S_\epsilon^s$ then $a(\xi)\eta_{\geq 1}(\xi) \in S_\epsilon^s$. Let

$$A_\epsilon^s = \{a(\xi)\eta_{\geq 1}(\xi) : a \in S_\epsilon^s\}. \tag{8.6}$$

Proposition 8.4. *Assume $a \in A_\epsilon^s$ and $s - \epsilon \geq 0$, then we have*

$$|E_1(u)| \lesssim \|u\|^2_{\dot H^{1/4-}\cap \dot H^{1/4+}} E_0(u).$$

Proof. Using the definition, we get

$$|E_1(u)| \leq \int_{\Gamma_4} |b_4(\xi_1,\xi_2,\xi_3,\xi_4)| \cdot |\widehat{u}(\xi_1)\widehat{u}(\xi_2)\widehat{u}(\xi_3)\widehat{u}(\xi_4)|.$$

From symmetries, we may assume that $|\xi_1| \leq |\xi_2| \leq |\xi_3| \leq |\xi_4|$. Localizing $|\xi_j| \sim N_j$ for N_j dyadic number, we may assume $N_3 \sim N_4 \gtrsim 1$. Then it follows from Proposition 8.3 that

$$\begin{aligned}
|E_1(u)| &\lesssim \sum_{N_j} \int_{\Gamma_4, |\xi_j|\sim N_j} {N_4}^{-1} |\widehat{u}(\xi_1)\widehat{u}(\xi_2) a(N_4)^{1/2}\widehat{u}(\xi_3) a(N_4)^{1/2}\widehat{u}(\xi_4)| \\
&\lesssim \sup_{N_4 \gtrsim 1} \sum_{N_1 \leq N_2 \leq N_4} N_4^{-1/2+\epsilon} {N_1}^{1/2} \|R_{N_1}u\|_2 \|R_{N_2}u\|_2 E_0(u) \\
&\lesssim \|u\|^2_{\dot H^{1/4-}\cap \dot H^{1/4+}} E_0(u).
\end{aligned}$$

Therefore we complete the proof of the proposition. ∎

Proposition 8.5. *Assume $a \in A_\epsilon^s$, $s - \epsilon \geq 0$ and $T \in (0,1]$. Then*

$$\left| \int_{-T}^{T} R_6(u) dt \right| \lesssim \|u\|^4_{F^{1/4-,1/4+}(T)} \|u\|^2_{F^{l,s}(T)}. \tag{8.7}$$

Proof. We first fix extension $\widetilde{u} \in C_0(\mathbb{R} : H^\infty)$ of u such that $\|R_k(\widetilde{u})\|_{F_k} \leq 2\|R_k(u)\|_{F_k(T)}$, $k \in \mathbb{Z}$. It suffices to prove that

$$\left| \int_0^T R_6(\widetilde{u}) dt \right| \lesssim \|\widetilde{u}\|^4_{F^{1/4-,1/4+}} \|\widetilde{u}\|^2_{F^{l,s}}. \tag{8.8}$$

For simplicity of the notations we still write $\widetilde{u} = u$. From symmetry, we get

$$\begin{aligned}
CR_6(u) \;\; = \;\; & \int_{\Gamma_6} [b_4(\xi_1,\xi_2,\xi_3,\xi_4+\xi_5+\xi_6)(\xi_4+\xi_5+\xi_6) \\
& -b_4(-\xi_4,-\xi_5,-\xi_6,\xi_4+\xi_5+\xi_6)(\xi_4+\xi_5+\xi_6)] \prod_{j=1}^6 \widehat{u}(\xi_j).
\end{aligned}$$

Localizing $|\xi_j| \sim N_j = 2^{k_j}$ and using symmetry, we may assume $N_1 \leq N_2 \leq N_3$, $N_4 \leq N_5 \leq N_6$ and $\max(N_j) \sim \mathrm{sub}(N_j) \gtrsim 1$ where $\max(N_j)$ and $\mathrm{sub}(N_j)$ are the maximum and second-maximum of $N_j, j = 1, 2, \ldots, 6$. Let $u_k = R_k(u)$ and $\xi_{456} = \xi_4 + \xi_5 + \xi_6$. Thus

$$\begin{aligned}
\left| \int_0^T R_6(u) dt \right| \;\; \lesssim \;\; & \sum_{N_j} \Big| \int_0^T \int_{\Gamma_6} [b_4(\xi_1,\xi_2,\xi_3,\xi_{456})(\xi_{456}) \\
& -b_4(-\xi_4,-\xi_5,-\xi_6,\xi_{456})(\xi_{456})] \prod_{j=1}^6 \widehat{u_{k_j}}(\xi_j) dt \Big|.
\end{aligned} \tag{8.9}$$

Let $\gamma : \mathbb{R}\to[0,1]$ denote a positive smooth function supported in $[-1,1]$ with the property that

$$\sum_{n\in\mathbb{Z}}\gamma^6(x-n)\equiv 1, \quad x\in\mathbb{R}.$$

We will bound (8.8) in several cases.

Case 1. $N_3\lesssim N_5, N_6$ and $N_5\sim N_6\gtrsim 1$. Then we get that the right-hand side of (8.9) is bounded by

$$\sum_{N_j}\sum_{|n|\lesssim 2^{k_6}}\Big|\int_{\mathbb{R}}\int_{\Gamma_6}[b_4(\xi_1,\xi_2,\xi_3,\xi_{456})-b_4(-\xi_4,-\xi_5,-\xi_6,\xi_{456})]$$
$$\xi_{456}[\gamma(2^{k_6}t-n)1_{[0,T]}(t)\widehat{u_{k_1}}(\xi_1)]\prod_{j=2}^{6}[\gamma(2^{k_6}t-n)\widehat{u_{k_j}}(\xi_j)]dt\Big|. \tag{8.10}$$

We observe first that

$$|A|=|\{n:\gamma(2^{k_6}t-n)1_{[0,T]}(t)\neq\gamma(2^{k_6}t-n),0,\ \forall\ t\in\mathbb{R}\}|\leq 4.$$

Let $f_{k_j}(\xi,\tau)=\mathcal{F}[\gamma(2^{k_6}t-n)R_ku]$, $j=1,2,\ldots,6$. Using proposition 8.3 and Plancherel's theorem, we easily get that the summation for $n\in A^c$ of (8.10) is bounded by

$$\sum_{N_j}\sum_{|n|\lesssim 2^{k_6},n\in A^c}(a(N_3)N_3^{-1}+a(N_6)N_6^{-1})N_3\left|\int_{\Gamma_6(\mathbb{R}^2)}\prod_{j=1}^{6}f_{k_j}(\xi_j,\tau_j)\right|. \tag{8.11}$$

Using Hölder's inequality and the embedding properties of B_k, we get

$$\begin{aligned}\left|\int_{\Gamma_6(\mathbb{R}^2)}\prod_{j=1}^{6}f_{k_j}(\xi_j,\tau_j)\right| &\lesssim \prod_{j=1}^{4}\|\mathcal{F}^{-1}(f_{k_j})\|_{L^4_xL^\infty_t}\\ &\quad\ \|\mathcal{F}^{-1}(f_{k_5})\|_{L^\infty_xL^2_t}\|\mathcal{F}^{-1}(f_{k_6})\|_{L^\infty_xL^2_t}\\ &\lesssim N_5^{-1}\prod_{j=1}^{4}N_j^{1/4}\prod_{j=1}^{6}\|f_{k_j}\|_{B_k}.\end{aligned}$$

Then we can bound (8.11) by

$$\begin{aligned}&\sum_{N_j}(a(N_3)N_3^{-1}+a(N_6)N_6^{-1})N_3\prod_{j=1}^{4}N_j^{1/4}\prod_{j=1}^{6}\|f_{k_j}\|_{F_k}\\ \lesssim\ &\|u\|^4_{F^{1/4-,1/4+}}\|u\|^2_{F^{l,s}},\end{aligned}$$

which is (8.8) as desired.

For the summation of $n\in A$, we observe that if $I\subset\mathbb{R}$ is an interval, $k\in\mathbb{Z}$, $f_k\in B_k$, and $f_k^I=\mathcal{F}(1_I(t)\cdot\mathcal{F}^{-1}(f_k))$ then

$$\sup_{j\in\mathbb{Z}_+}2^{j/2}\|\eta_j(\tau-\omega(\xi))\cdot f_k^I\|_{L^2}\lesssim\|f_k\|_{B_k}.$$

Let $f_{k_i,j_i}=\eta_{j_i}(\tau-\omega(\xi))\mathcal{F}[\gamma(2^{k_6}t-n)R_ku]$, $i=1,2,\ldots,5$, and $f_{k_6,j_6}=\eta_{j_6}(\tau-\omega(\xi))\mathcal{F}[\gamma(2^{k_6}t-n)1_{[0,T]}(t)R_ku]$. If $j_6\geq 100k_6$, then by checking the support properties we get $\int_{\Gamma_6(\mathbb{R}^2)}\prod_{i=1}^{6}f_{k_i,j_i}(\xi_j,\tau_j)\equiv 0$ unless $|j_{max}-j_{sub}|\leq 10$ and $j_{max}\geq 100k_6$,

where j_{max} and j_{sub} are the maximum and sub-maximum of $j_1, j_2, \ldots, j_6$. By using Cauchy-Schwarz inequality, we get that

$$\begin{aligned}
&\sum_{j_i} \left| \int_{\Gamma_6(\mathbb{R}^2)} \prod_{i=1}^{6} f_{k_i,j_i}(\xi_j, \tau_j) \right| \\
\lesssim\ & 2^{(k_1+\ldots+k_6)/2} 2^{(j_1+\ldots+j_6)/2} 2^{-(j_{max})} \prod_{i=1}^{6} \|f_{k_i,j_i}\|_{L^2} \\
\lesssim\ & N_5^{-1} \prod_{j=1}^{4} N_j^{1/4} \prod_{j=1}^{6} \|f_{k_j}\|_{F_k},
\end{aligned}$$

which is acceptable. If $j_6 \leq 100k_6$ then we argue as before for $n \in A^c$, hence we get that

$$\sum_{j_i} \left| \int_{\Gamma_6(\mathbb{R}^2)} \prod_{i=1}^{6} f_{k_i,j_i}(\xi_j, \tau_j) \right| \lesssim k_6 N_5^{-1} \prod_{j=1}^{4} N_j^{1/4} \prod_{j=1}^{6} \|f_{k_j}\|_{F_k},$$

which combined with (8.11) gives (8.8).

Case 2. $N_6 \lesssim N_2, N_3$ and $N_2 \sim N_3 \gtrsim 1$. From symmetry, this case is identical to Case 1. We omit the details.

Case 3. $N_2, N_5 \ll N_3, N_6$ and $N_6 \sim N_3 \gtrsim 1$. From the proof of Proposition 8.3, we get that

$$|b_4(\xi_1, \xi_2, \xi_3, \xi_{456}) - b_4(-\xi_4, -\xi_5, -\xi_6, \xi_{456})| \cdot |\xi_{456}| \lesssim a(N_3) + a(N_6) N_6^{-1}.$$

Then following the same argument as in Case 1, we obtain (8.8) as desired. ■

Lemma 8.6 (Lemma 5.5, [26]). *There is a sequence $\{\beta_\lambda\}$ with the following properties:*

(a) $\lambda^{2s} \|P_\lambda(u_0)\|_{L^2}^2 \leq \beta_\lambda \|u_0\|_{H^s}^2$,

(b) $\sum \beta_\lambda \lesssim 1$,

(c) β_λ is slowly varying in the sense that

$$|\log_2 \beta_\lambda - \log_2 \beta_\mu| \leq \frac{\epsilon}{2} |\log_2 \lambda - \log_2 \mu|.$$

Proof of Proposition 8.1. In view of the definition, we get

$$\|u\|_{E^{l,s}(T)}^2 \lesssim \|P_{\leq 0} u_0\|_{\dot{H}^l}^2 + \sum_{k \geq 1} \sup_{t \in [-T,T]} 2^{2ks} \|P_k(u(t))\|_{L^2}^2. \tag{8.12}$$

We will prove that if $k \geq 1$ then

$$\sup_{t \in [-T,T]} 2^{2ks} \|P_k(u(t))\|_{L^2}^2 \lesssim \beta_k (\|P_{\geq 1} u_0\|_{H^s}^2 + \|u\|_{F^{\frac{1}{4}-, \frac{1}{4}+}(T)}^4 \cdot \|u\|_{F^{l,s}(T)}^2), \tag{8.13}$$

which suffices to prove Proposition 8.1 in view of Lemma 8.6 (b). In order to prove (8.13) for some fixed k_0 we define the sequence

$$a_k = 2^{2ks} \max(1, \beta_{k_0}^{-1} 2^{-\epsilon|k-k_0|}).$$

Using the slowly varying condition (iii), then we get

$$\begin{aligned}\sum_{k\geq 1} a_k\|P_k(u_0)\|_{L^2}^2 &\lesssim \sum_k 2^{2ks}\|P_k(u_0)\|_{L^2}^2\\ &\quad +2^{-\epsilon|k-k_0|/2}2^{2ks}\beta_k^{-1}\|P_k(u_0)\|_{L^2}^2\\ &\lesssim \|P_{\geq 1}(u_0)\|_{H^s}^2.\end{aligned}$$

We may assume that $\beta_0=1$. Then we see that $\max(|\beta_k|,|\beta_k^{-1}|)\leq 2^{k\epsilon/2}$. Correspondingly we find a function $a(\xi)\in S_\epsilon^s$ so that

$$a(\xi)\sim a_k,\quad |\xi|\sim 2^k.$$

Thus we apply Proposition 8.4, 8.5 for $a(\xi)\eta_{\geq 1}(\xi)$, then we get

$$\sup_{t\in[-T,T]}|E_0(u(t)+E_1(u(t))|\leq |E_0(u_0)+E_1(u_0)|+\left|\int_{-T}^T R_6(u)dt\right|,$$

from which we see that

$$\sup_{t\in[-T,T]}|E_0(u(t)|\leq |E_0(u_0)|+\|u\|_{F^{1/4-,1/4+}(T)}^4\|u\|_{F^{l,s}(T)}^2.$$

Therefore, we get

$$\left(\sum_{k\geq 1} a_k\|P_k(u(t))\|_{L^2}^2\right)\lesssim \|P_{\geq 1}u_0\|_{H^s}^2+\|u\|_{F^{1/4-,1/4+}(T)}^4\|u\|_{F^{l,s}(T)}^2,$$

which at $k=k_0$ gives (8.13) as desired. ■

9. Proof of Theorem 1.3

In this section we devote to prove Theorem 1.3. The main ingredients are energy estimates and short-time trilinear estimates. The idea is due to Ionescu, Kenig and Tataru [16].

Proposition 9.1. *Let $l,s\geq 0$, $T\in(0,1]$, and $u\in F^{l,s}(T)$, then*

$$\sup_{t\in[-T,T]}\|u(t)\|_{\dot H^l\cap \dot H^s}\lesssim \|u\|_{F^{l,s}(T)}. \tag{9.1}$$

Proof. In view of the definitions, it suffices to prove that if $k\in\mathbb{Z}$, $t_k\in[-1,1]$, and $\widetilde{u}_k\in F_k$ then

$$\|\mathcal{F}[\widetilde{u}_k(t_k)]\|_{L_\xi^2}\lesssim \|\mathcal{F}[\widetilde{u}_k\cdot\eta_0(2^{k_+}(t-t_k))]\|_{B_k}. \tag{9.2}$$

Let $f_k=\mathcal{F}[\widetilde{u}_k\cdot\eta_0(2^{k_+}(t-t_k))]$, so

$$\mathcal{F}[\widetilde{u}_k(t_k)](\xi)=c\int_{\mathbb{R}} f_k(\xi,\tau)e^{it_k\tau}d\tau.$$

From the definition of B_k, we get that

$$\|\mathcal{F}[\widetilde{u}_k(t_k)]\|_{L_\xi^2}\lesssim \left\|\int_{\mathbb{R}}|f_k(\xi,\tau)|d\tau\right\|_{L_\xi^2}\lesssim \|f_k\|_{B_k},$$

which completes the proof of the proposition. ■

Proposition 9.2. *Assume $T \in (0,1]$, $u, v \in C([-T,T] : H^\infty)$ and*

$$u_t + \mathcal{H}u_{xx} = v \text{ on } \mathbb{R}^2 \times (-T,T). \tag{9.3}$$

Then for any $l, s \geq 0$,

$$\|u\|_{F^{l,s}(T)} \lesssim \|u\|_{E^{l,s}(T)} + \|v\|_{N^{l,s}(T)}. \tag{9.4}$$

Proof. In view of the definitions, we see that the square of the right-hand side of (9.4) is equivalent to

$$\begin{aligned}&\sum_{k\leq 0}\big(2^{2lk}\|R_k(u(0))\|_{L^2}^2 + 2^{2lk}\|R_k(v)\|_{N_k(T)}^2\big)\\ &+\sum_{k\geq 1}\big(\sup_{t_k\in[-T,T]} 2^{2sk}\|R_k(u(t_k))\|_{L^2}^2 + 2^{2sk}\|R_k(v)\|_{N_k(T)}^2\big).\end{aligned} \tag{9.5}$$

Thus, from definitions, it suffices to prove that if $k \in \mathbb{Z}$ and $u, v \in C([-T,T] : H^\infty)$ solve (9.3), then

$$\begin{aligned}&\|R_k(u)\|_{F_k(T)} \lesssim \|R_k(u(0))\|_{L^2} + \|R_k(v)\|_{N_k(T)} \text{ if } k \leq 0;\\ &\|R_k(u)\|_{F_k(T)} \lesssim \sup\nolimits_{t_k\in[-T,T]} \|R_k(u(t_k))\|_{L^2} + \|R_k(v)\|_{N_k(T)} \text{ if } k \geq 1.\end{aligned} \tag{9.6}$$

Let $\widetilde{v}$ denote an extension of $R_k(v)$ such that $\|\widetilde{v}\|_{N_k} \leq C\|v\|_{N_k(T)}$. Using (3.13), we may assume that $\widetilde{v}$ is supported in $\mathbb{R} \times [-T - 2^{-k_+-10}, T + 2^{-k_+-10}]$, $k \in \mathbb{Z}$. Indeed, let $\beta(t)$ be a smooth function such that

$$\beta(t) = 1, \text{ if } t \geq 1; \quad \beta(t) = 0, \text{ if } t \leq 0.$$

Thus $\beta(2^{k_++10}(t + T + 2^{-k_+-10}))$, $\beta(-2^{k_++10}(t - T - 2^{-k_+-10})) \in S_k$. Then we see that $\beta(2^{k_++10}(t + T + 2^{-k_+-10}))\beta(-2^{k_++10}(t - T - 2^{-k_+-10}))$ is supported in $[-T - 2^{-k_+-10}, T + 2^{-k_+-10}]$, and equal to 1 in $[-T,T]$. For $t \geq T$ we define

$$\widetilde{u}(t) = \eta_0(2^{k_++5}(t-T))[W(t-T)R_k(u(T)) + \int_T^t W(t-s)(R_k(\widetilde{v}(s)))ds].$$

For $t \leq -T$ we define

$$\widetilde{u}(t) = \eta_0(2^{k_++5}(t+T))[W(t+T)R_k(u(-T)) + \int_{-T}^t W(t-s)(R_k(\widetilde{v}(s)))ds].$$

For $t \in [-T,T]$ we define $\widetilde{u}(t) = u(t)$. It is clear that $\widetilde{u}$ is an extension of u. Also, using (3.13), we get

$$\|u\|_{F_k(T)} \lesssim \sup_{t_k\in[-T,T]} \|\mathcal{F}[\widetilde{u}\cdot\eta_0(2^{k_+}(t-t_k))]\|_{B_k}. \tag{9.7}$$

Indeed, to prove (9.7), it suffices to prove that

$$\sup_{t_k\in\mathbb{R}} \|\mathcal{F}[\widetilde{u}\cdot\eta_0(2^{k_+}(t-t_k))]\|_{B_k} \lesssim \sup_{t_k\in[-T,T]} \|\mathcal{F}[\widetilde{u}\cdot\eta_0(2^{k_+}(t-t_k))]\|_{B_k}. \tag{9.8}$$

For $t_k > T$, since $\widetilde{u}$ is supported in $[-T - 2^{-k_+-5}, T + 2^{-k_+-5}]$, it is easy to see that

$$\widetilde{u}\eta_0(2^{k_+}(t-t_k)) = \widetilde{u}\eta_0(2^{k_+}(t-T))\eta_0(2^{k_+}(t-t_k)).$$

Therefore, we get from (3.11) that

$$\sup_{t_k>T} \|\mathcal{F}[\widetilde{u}\cdot\eta_0(2^{k_+}(t-t_k))]\|_{B_k} \lesssim \sup_{t_k\in[-T,T]} \|\mathcal{F}[\widetilde{u}\cdot\eta_0(2^{k_+}(t-t_k))]\|_{B_k}.$$

Using the same way for $t_k < -T$, we obtain (9.7) as desired.

It remains to prove (9.6). In view of the definitions, (9.7) and (3.11), it suffices to prove that if $k \in \mathbb{Z}$, $\phi_k \in L^2$ with $\widehat{\phi_k}$ supported in I_k, and $v_k \in N_k$ then

$$\|\mathcal{F}[u_k \cdot \eta_0(2^{k_+}t)]\|_{B_k} \lesssim \|\phi_k\|_{L^2} + \|(\tau - \omega(\xi) + i2^{k_+})^{-1} \cdot \mathcal{F}(v_k)\|_{B_k}, \tag{9.9}$$

where

$$u_k(t) = W(t)(\phi_k) + \int_0^t W(t-s)(v_k(s))ds. \tag{9.10}$$

Straightforward computations show that

$$\begin{aligned}&\mathcal{F}[u_k \cdot \eta_0(2^{k_+}t)](\xi,\tau) = \widehat{\phi_k}(\xi) \cdot 2^{-k_+}\widehat{\eta_0}(2^{-k_+}(\tau - \omega(\xi)))\\ &+C\int_{\mathbb{R}} \mathcal{F}(v_k)(\xi,\tau') \cdot \frac{2^{-k_+}\widehat{\eta_0}(2^{-k_+}(\tau-\tau')) - 2^{-k_+}\widehat{\eta_0}(2^{-k_+}(\tau-\omega(\xi)))}{\tau' - \omega(\xi)} d\tau'.\end{aligned}$$

We observe now that

$$\begin{aligned}&\left|\frac{2^{-k_+}\widehat{\eta_0}(2^{-k_+}(\tau-\tau')) - 2^{-k_+}\widehat{\eta_0}(2^{-k_+}(\tau-\omega(\xi)))}{\tau' - \omega(\xi)} \cdot (\tau' - \omega(\xi) + i2^{k_+})\right|\\ &\lesssim 2^{-k_+}(1+2^{-k_+}|\tau - \tau'|)^{-4} + 2^{-k_+}(1+2^{-k_+}|\tau-\omega(\xi)|)^{-4}.\end{aligned}$$

Using (3.9) and (3.10), we complete the proof of the proposition. ■

We prove a crucial trilinear estimates in the following proposition.

Proposition 9.3. *Let $0 \le l \le 1/4$ and $s \ge 1/4$. Then*

$$\begin{aligned}\|\partial_x(uvw)\|_{N^{l,s}(T)} \lesssim\ & \|u\|_{F^{l,s}(T)}\|v\|_{F^{1/4,1/4}(T)}\|w\|_{F^{1/4,1/4}(T)}\\ &+\|u\|_{F^{1/4,1/4}(T)}\|v\|_{F^{l,s}(T)}\|w\|_{F^{1/4,1/4}(T)}\\ &+\|u\|_{F^{1/4,1/4}(T)}\|v\|_{F^{1/4,1/4}(T)}\|w\|_{F^{l,s}(T)}.\end{aligned} \tag{9.11}$$

Proof. Since $R_kR_j = 0$ if $k \ne j$, then we can fix extensions $\widetilde{u}, \widetilde{v}, \widetilde{w}$ of u, v, w such that $\|R_k(\widetilde{u})\|_{F_k} \le 2\|R_k(u)\|_{F_k(T)}$, $\|R_k(\widetilde{v})\|_{F_k} \le 2\|R_k(v)\|_{F_k(T)}$ and $\|R_k(\widetilde{w})\|_{F_k} \le 2\|R_k(w)\|_{F_k(T)}$ for any $k \in \mathbb{Z}$. In view of definition, we get

$$\begin{aligned}\|\partial_x(\widetilde{u}\widetilde{v}\widetilde{w})\|_{N^{l,s}}^2 =\ & \sum_{k_4=-\infty}^{-1} 2^{2lk_4}\|R_{k_4}(\partial_x(\widetilde{u}\widetilde{v}\widetilde{w}))\|_{N_{k_4}}^2\\ &+\sum_{k_4=0}^{\infty} 2^{2sk_4}\|R_{k_4}(\partial_x(\widetilde{u}\widetilde{v}\widetilde{w}))\|_{N_{k_4}}^2.\end{aligned}$$

Let $\widetilde{u}_k = R_k(\widetilde{u})$, $\widetilde{v}_k = R_k(\widetilde{v})$ and $\widetilde{w}_k = R_k(\widetilde{u})$. Then we get

$$\|R_{k_4}(\partial_x(\widetilde{u}\widetilde{v}\widetilde{w}))\|_{N_{k_4}} \lesssim \sum_{k_1,k_2,k_3\in\mathbb{Z}} \|R_{k_4}(\partial_x(\widetilde{u}_{k_1}\widetilde{v}_{k_2}\widetilde{w}_{k_3}))\|_{N_{k_4}}.$$

From symmetry we may assume $k_1 \le k_2 \le k_3$. Dividing the summation into several parts, we get

$$\sum_{k_1\le k_2\le k_3} \|R_{k_4}(\partial_x(\widetilde{u}_{k_1}\widetilde{v}_{k_2}\widetilde{w}_{k_3}))\|_{N_{k_4}} \le \sum_{j=1}^{6}\sum_{\{k_i\}\in A_j} \|R_{k_4}(\partial_x(\widetilde{u}_{k_1}\widetilde{v}_{k_2}\widetilde{w}_{k_3}))\|_{N_{k_4}}, \tag{9.12}$$

where A_j, $j = 1, 2, \ldots, 6$, are as in the proof of Proposition 6.3. We will apply Proposition 7.1-7.6 obtained in the Section 7 to bound the six terms in (9.12). For

example, for the first term, from Proposition 7.1, we have

$$\begin{aligned}
&\Big\|2^{sk_4}\sum_{\{k_i\}\in A_1}\|R_{k_4}(\partial_x(\widetilde{u}_{k_1}\widetilde{v}_{k_2}\widetilde{w}_{k_3}))\|_{N_{k_4}}\Big\|_{l^2_{k_4}}\\
\leq\ & C\Big\|2^{sk_4}\sum_{k_i\in A_1}\min(2^{k_1/2},|k_2|+1)\|\widetilde{u}_{k_1}\|_{F_{k_1}}\|\widetilde{v}_{k_2}\|_{F_{k_2}}\|\widetilde{w}_{k_3}\|_{F_{k_3}}\Big\|_{l^2_{k_4}}\\
\leq\ & \|\widetilde{u}\|_{F^{1/4,1/4}}\|\widetilde{v}\|_{F^{1/4,1/4}}\|\widetilde{w}\|_{F^{l,s}}.
\end{aligned}$$

For the other terms we can handle them in the similar ways. Therefore we complete the proof of the proposition. ■

We prove now Theorem 1.3. Fix $0<l<1/4$ and $s>1/4$. By the scaling (1.4) we may assume that

$$\|\phi\|_{\dot{H}^l\cap\dot{H}^s}\leq \delta_0/M, \tag{9.13}$$

where $M\gg 1$ and δ_0 is given as in Proposition 8.1. For any $T'\in[0,1]$, we denote $X(T')=\|u\|_{E^{l,s}(T')}+\|\partial_x(u^3)\|_{N^{l,s}(T')}$. We assume first that $X(T')$ is continuous and satisfies

$$\lim_{T'\to 0}X(T')\lesssim\|\phi\|_{\dot{H}^l\cap\dot{H}^s}. \tag{9.14}$$

We will prove that

$$X(T')\leq 2c\delta_0/M,\qquad \text{for any } T'\in[0,1]. \tag{9.15}$$

By using bootstrap (e.g. see [36]), we may assume that $X(T')\leq 3c\delta_0/M$ for any $T'\in[0,1]$. It follows from Proposition 9.2, 9.3 that for any $T'\in[0,1]$ we have

$$\begin{cases}\|u\|_{F^{l,s}(T')}\lesssim\|u\|_{E^{l,s}(T')}+\|\partial_x(u^3)\|_{N^{l,s}(T')};\\ \|\partial_x(u^3)\|_{N^{l,s}(T')}\lesssim\|u\|^3_{F^{l,s}(T')};\\ \|u\|^2_{E^{l,s}(T')}\lesssim\|\phi\|^2_{\dot{H}^l\cap\dot{H}^s}+\|u\|^6_{F^{l,s}(T')}.\end{cases} \tag{9.16}$$

Thus we get

$$X(T')^2\lesssim\|\phi\|^2_{\dot{H}^l\cap\dot{H}^s}+X(T')^6, \tag{9.17}$$

from which and (9.13) and the assumption $X(T')\leq 3c\delta_0/M$, we obtain (9.15) as desired. Then, using (9.16), (9.15) and Proposition 9.1, we have

$$\|u\|_{C([-1,1];\dot{H}^l\cap\dot{H}^s)}\lesssim\|\phi\|_{\dot{H}^l\cap\dot{H}^s}. \tag{9.18}$$

For general ϕ and the L^2 norm of the solution, we just use the scaling (1.4) and the L^2 conservation law (1.6).

It remains to prove that $X(T)$ is continuous and (9.14). Obviously, for $u\in C([-T,T]:H^\infty)$ the first component $T'\to\|u\|_{E^{l,s}(T')}$ is increasing and continuous on $[-T,T]$ and

$$\lim_{T'\to 0}\|u\|_{E^{l,s}(T')}\lesssim\|\phi\|_{\dot{H}^l\cap\dot{H}^s}.$$

For the second component it follows from the similar argument as in the proof of Lemma 4.2 in [16]. We omit the details.

Remark 9.4. From the proof we see that we actually prove a stronger result than that stated in Theorem 1.3. We expect that some wellposedness results hold for the initial data in $\dot{H}^l\cap\dot{H}^s$.

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